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Symmetry structure of special geometries

B. de Wit[1]
Institute for Theoretical Physics, University of Utrecht
Princetonplein 5, 3508 TA Utrecht, The Netherlands

F. Vanderseypen[2] and **A. Van Proeyen**[3]
Instituut voor theoretische fysica
Universiteit Leuven, B-3001 Leuven, Belgium

ABSTRACT

Using techniques from supergravity and dimensional reduction, we study the full isometry algebra of Kähler and quaternionic manifolds with special geometry. These two varieties are related by the so-called **c** map, which can be understood from dimensional reduction of supergravity theories or by changing chirality assignments in the underlying superstring theory. An important subclass, studied in detail, consists of the spaces that follow from real special spaces using the so-called **r** map. We generally clarify the presence of 'extra' symmetries emerging from dimensional reduction and give the conditions for the existence of 'hidden' symmetries. These symmetries play a major role in our analysis. We specify the structure of the homogeneous special manifolds as coset spaces G/H. These include all homogeneous quaternionic spaces.

[1] On leave of absence from the Institute for Theoretical Physics, Utrecht, The Netherlands; E-mail BDEWIT at FYS.RUU.NL

[2] Bitnet FGBDA46 at BLEKUL11

[3] Onderzoeksleider, N.F.W.O. Belgium; Bitnet FGBDA19 at BLEKUL11

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1 Introduction

Supersymmetry is an important ingredient in many of the theories that attempt to unify the fundamental forces between elementary particles. At energies that are low compared to the Planck scale, the relevant effective theories are expected to take the form of supergravity coupled to matter. Superstrings, which at present provide the best candidate for a completely unified theory, allow for a huge number of classical vacua. The ones that are considered phenomenologically viable are compactifications of the heterotic string on Calabi-Yau manifolds (or on the possibly more general class of $(2,2)$ superconformal theories with $c=9$) and lead to four-dimensional theories with $N=1$ space-time supersymmetry [1]. For such compactifications $N=2$ supergravity is relevant, as these superconformal theories can also be used in order to compactify type-II strings; therefore many features of these compactifications can be understood on the basis of $N=2$ space-time supersymmetry, irrespective of the fact that this symmetry is not realized for the full effective low-energy theory corresponding to the heterotic string [2, 3, 4]. In this context the coupling of vector multiplets to $N=2$ supergravity is particularly important. Like in many supergravity theories the spinless matter fields in these theories parametrize a non-linear sigma model. In this case these sigma models define Kähler manifolds whose structure is encoded in a single holomorphic and homogeneous function [5]. The underlying geometry of such manifolds is called *special* [6, 7]. When the effective supergravity theory that emerges from superstring compactifications has spinless fields whose potential is flat, their unconstrained vacuum-expectation values parametrize possible superstring ground states. Therefore the moduli space of $(2,2)$ superconformal field theories with $c=9$ (and likewise the moduli spaces of Calabi-Yau three-folds) exhibits special geometry, and this plays an important role in the study of these manifolds and intriguing phenomena such as mirror manifolds and generalized target-space duality (see, for instance, [6, 8, 9, 10] and references therein).

Special geometry is not confined to Kähler spaces. In $N=2$ supergravity one can also have scalar supermultiplets, consisting of spin-0 and spin-$\frac{1}{2}$ fields. In this case the spinless fields lead to quaternionic non-linear sigma models [11]. There is a subclass of such models whose structure is again encoded in a single holomorphic and homogeneous function. These spaces emerge also in $(2,2)$ compactifications of type-II superstrings and are called *special* quaternionic spaces. In this way both a special Kähler and a special quaternionic manifold are assigned to the same function. When comparing the compactification of the type-IIA superstring to the compactification on the same superconformal system of the type-IIB superstring, one discovers that the functions characterizing the Kähler and the quaternionic space are interchanged [2]. Hence there is a mapping, the so-called **c** map, that is induced by changing the superstring from type IIA to IIB, and vice versa [3]. Within the context of supergravity alone, it turns out that the **c** map is induced by dimensional reduction of $d=4$ supergravity coupled to vector multiplets. The resulting $d=3$ supergravity theory couples only to quaternionic non-linear sigma models. Supersymmetry, which is preserved by dimensional reduction, thus ensures that every special Kähler space is mapped to a quaternionic space. Note, however, that there exist quaternionic spaces that couple to $N=2$ supergravity, but are not *special*, i.e., they are not in the image of the **c** map.

In this paper we study the isometry structure of manifolds with special geometry.

Because of supersymmetry, the isometry transformations usually define invariances of the full supergravity theories, provided they are suitably extended to act on vector and spinor fields. In $d = 4$ dimensions such transformations often take the form of generalized duality invariances, which leave the field equations, but not the Lagrangian, invariant [12]. It turns out to be convenient to introduce also a real version of special manifolds, corresponding to the non-linear sigma models that one obtains in $d = 5$ space-time dimensions when coupling $N = 2$ supergravity to vector multiplets [13]. Upon reduction to $d = 4$ space-time dimensions, these theories give rise to special Kähler manifolds, so that one can again consider a mapping, but now from the special real to the special Kähler manifolds. This leads to the definition of the **r** map. Not all special Kähler manifolds are contained in the image of the **r** map. (Following [14] we sometimes call the special real spaces and their Kähler and quaternionic counterparts, 'very special' spaces.) We thus encounter the following relation between the real, Kähler and quaternionic special spaces,

$$\mathbb{R}_{n-1} \xrightarrow{\mathbf{r}} \mathbb{C}_n \ , \qquad \mathbb{C}_n \xrightarrow{\mathbf{c}} \mathbb{H}_{n+1} \ , \tag{1.1}$$

where $n-1$, n and $n+1$ denote the real, complex and quaternionic dimension of the real, Kähler and quaternionic spaces, respectively. The special Kähler spaces in the image of the **r** map were already studied in [15], because they allow supergravity couplings with flat potentials, and are determined by a symmetric three-index tensor d_{ABC}. Under mild assumptions one can show that the corresponding couplings of vector multiplets arise from compactifications of type-IIA superstring theories. Furthermore nearly all homogeneous special Kähler and quaternionic spaces are in this subclass [16, 17]. For a preliminary account of our results, see [18].

The fact that the **r** and the **c** map are induced by dimensional reduction has intriguing implications for the isometry structure of the special geometries that are in the image of (one of) these maps. First of all, the dimension of the non-linear sigma models is increased by spinless fields that originate from the higher-dimensional tensor and vector fields. Secondly, part of the gauge and general-coordinate transformations pertaining to the extra space-time coordinate(s) remain as symmetries after reduction and lead to an enlargement of the isometry group of the non-linear sigma model. An extra feature is present in the dimensional reduction to three dimensions, where abelian vector fields are converted into scalars, which are subject to again extra symmetries. A more detailed analysis shows that, upon application of the map, the rank of the space is increased by precisely one unit and the roots associated to the extra isometries are confined to a particular half-space of the root lattice corresponding to the full algebra of isometry transformations. In addition there may exist so-called *hidden* symmetries, whose existence cannot be inferred directly from the higher-dimensional origin of the theory (note that we deviate here from the terminology used in the supergravity literature, where both "extra" and "hidden" symmetries are called hidden). The roots corresponding to these hidden symmetries also take characteristic positions in the root lattice. We already presented an analysis of the isometry structure of the special quaternionic manifolds in [19]; the results for the special Kähler spaces in the image of the **r** map follow from the analysis presented in [15]. Some characteristic features of the extra symmetries were independently noted in [20].

The possible existence of the hidden symmetries is analyzed in detail, first for the generic special quaternionic spaces, and subsequently for the Kähler and quaternionic spaces that are in the image of the **r** map and the $\mathbf{c}\circ\mathbf{r}$ map. The homogeneous spaces play a special role, because under certain conditions the homogeneity of the space is preserved by the **r** and the **c** map. For homogeneous spaces the isometries act transitively on the manifold so that every two points are related by an element of the isometry group. The orbit swept out by the action of the isometry group G from any given point is (locally) isomorphic to the coset space G/H, where H is the isotropy group of that point. For non-compact homogeneous spaces where H is the maximal compact subgroup of G, there exists a solvable subgroup that acts transitively, whose dimension is equal to the dimension of the space. Such spaces are called *normal*. Some time ago a general classification of normal quaternionic spaces[1], i.e. quaternionic spaces that admit a solvable transitive group of isometries, was given [21].

All the normal quaternionic spaces, with the exception of the quaternionic projective spaces, are in the image of the **c** map, as was demonstrated explicitly in [16]. In [17] two of us gave a classification of the homogeneous Kähler and quaternionic spaces that are in the image of the $\mathbf{c}\circ\mathbf{r}$ map, containing the above spaces as well as certain homogeneous quaternionic and Kähler spaces that were overlooked in the classification of [21]. This class of spaces is characterized by realizations of real Clifford algebras with positive-definite metric. For every (not necessarily irreducible) realization of a real Clifford algebra with positive signature and arbitrary dimension, denoted by $q+1$ (so that $q \geq -1$), there exists an homogeneous real, an homogeneous special Kähler, and an homogeneous special quaternionic manifold.

Alekseevskiĭ presented the solvable subalgebra of the quaternionic isometry groups, which completely determines the homogeneous spaces. The solvable algebra does not contain the hidden symmetries discussed above, but when the space is symmetric, the hidden symmetries complete the solvable algebra to a simple algebra. In this paper we determine the hidden symmetries for the homogeneous spaces. In this way we are able to determine the isometry group G (which is not semisimple for the non-symmetric spaces) and its compact isotropy subgroup H for all the homogeneous quaternionic spaces and the homogeneous Kähler spaces in the image of the **r** map. For the real special spaces we have similar results, but they are somewhat less general as our methods restrict us to isometries that are symmetries of the full $d=5$ supergravity theory containing a non-linear sigma model with the real special space as its target space. For the non-symmetric spaces, the Cartan subalgebra of the isometry group G contains one generator with respect to which all the generators have eigenvalues equal to 0, 1 or 2. Apart from this generator, the generators with eigenvalue 0 define a semisimple group, whose maximal compact subgroup is equal to the isotropy group H. This semisimple group is $\mathcal{O}\otimes\mathcal{S}_q$, where $\mathcal{O}$ equals $SO(q{+}1,1)$, $SO(q{+}2,2)$ or $SO(q{+}3,3)$ for the real, Kähler, and quaternionic case, respectively, and $\mathcal{S}_q$ is the (compact) metric-preserving group of transformations in the spinor space that commute with the Clifford algebra. Furthermore the generators with eigenvalue 1 form a $(spinor, vector)$ real representation with respect to $\mathcal{O}\otimes\mathcal{S}_q$. There are no generators with eigenvalue 2 in the real case. For the Kähler case there is just one such generator, while for the quaternionic case these generators constitute a $(vector, singlet)$ representation.

This paper is organized as follows. In section 2 we introduce Kähler, real and quater-

[1] According to a conjecture [21] the homogeneous quaternionic spaces consist of the normal and the compact symmetric ones.

nionic special geometry and give the corresponding supergravity Lagrangians with particular attention to the isometry structure of manifolds that are related by the **r** or the **c** map. We formulate the conditions for the presence of the hidden symmetries and describe the characteristic root-lattice decompositions that emerge upon application of one of these maps. Section 3 describes the main features and some of their consequences of the isometry algebra of the generic Kähler and quaternionic special manifolds. Furthermore we analyze under which conditions the **c** map (or the **r** map) preserves the homogeneity of the manifolds. In section 4 we study the special real manifolds and their corresponding Kähler and quaternionic spaces and exhibit their symmetry structure. Section 5 contains a discussion of the subclass of these spaces that are homogeneous, deriving the group of isometries and isotropies, described in the previous paragraph.

Some of the more technical details have been relegated to appendices. In appendix A we discuss some features related to the role played by solvable algebras in the theory of non-compact homogeneous spaces. Appendix B contains some useful formulae related to special Kähler spaces. In appendix C we explain the symplectic reparametrizations of special Kähler manifolds, which encompass the so-called generalized duality invariances, and give the transformation rules of the scalar, spinor and vector fields.

2 Preliminaries

In this section we introduce the special geometries that appear in non-linear sigma models coupled to $N=2$ supergravity. We give the relevant supergravity Lagrangians, which, via dimensional reduction, give rise to or originate from $N=2$ supergravity coupled to n vector supermultiplets in $d=4$ space-time dimensions. As explained in the introduction, dimensional reduction induces the **c** and **r** maps that relate the various types of special geometries associated with the non-linear sigma models of the supergravity theories. We start by summarizing the situation for the four-dimensional theory, at least as far as the bosonic part of its Lagrangian is concerned. The corresponding non-linear sigma models define the *special* Kähler manifolds. Then we discuss how a sub-class of the $d=4$ theories originate from supergravity in $d=5$ dimensions, whose non-linear sigma models are based on *special* real manifolds. Subsequently we give the relation with supergravity in $d=3$ dimensions, which lead to the *special* quaternionic manifolds. Throughout this section we try to exhibit the relation between the various non-linear sigma models with particular emphasis on the structure of their isometry groups.

2.1 Special Kähler manifolds

The coupling of n vector multiplets to $N=2$ supergravity in $d=4$ dimensions is encoded in a single holomorphic function $F(X)$, which is homogeneous of second degree in terms of the $n+1$ variables X^I labeled by indices $I=0,1,\ldots,n$. Therefore it satisfies identities such as $F=\frac{1}{2}F_I X^I$, $F_I = F_{IJ}X^J$, $X^I F_{IJK}=0$, where the subscripts $I,J,\ldots$ denote differentiation with respect to X^I, X^J, etc. The bosonic Lagrangian reads [5, 22]

$$\begin{aligned} e^{-1}\mathcal{L} &= -\tfrac{1}{2}R + (XN\bar{X})^{-1}\,\mathcal{M}_{I\bar{J}}\,\partial_\mu X^I\,\partial^\mu \bar{X}^J \\ &\quad +\tfrac{1}{4}\left\{\mathcal{N}_{IJ}\,F^{+I}_{\mu\nu}\,F^{+\mu\nu J} + \bar{\mathcal{N}}_{IJ}\,F^{-I}_{\mu\nu}\,F^{-\mu\nu J}\right\}, \end{aligned} \tag{2.1}$$

where R is the Ricci scalar, $F^{\pm I}_{\mu\nu}$ are the (anti)selfdual components of the $n+1$ field strengths and the tensors N_{IJ}, $\mathcal{M}_{I\bar{J}}$ and $\mathcal{N}_{IJ}$ are defined by

$$\begin{aligned} N_{IJ} &= \frac{1}{4}\Big(F_{IJ}+\bar{F}_{IJ}\Big)\,, \\ \mathcal{M}_{I\bar{J}} &= N_{IJ} - \frac{(N\bar{X})_I\,(NX)_J}{XN\bar{X}}, \\ \mathcal{N}_{IJ} &= \frac{1}{4}\bar{F}_{IJ} - \frac{(NX)_I\,(NX)_J}{XNX}. \end{aligned} \tag{2.2}$$

Here we used an obvious notation where $(NX)_I = N_{IJ}X^J$, $XN\bar{X} = X^I N_{IJ}\bar{X}^J$, etc..

The Lagrangian (2.1) contains $n+1$ vector fields, with one of them (the so-called graviphoton) belonging to the $N=2$ supergravity multiplet. However, it depends on only n scalar fields, because $\mathcal{M}_{I\bar{J}}$ has a null vector proportional to X^I and $\bar{X}^J$, as one easily verifies, while the tensors $\mathcal{M}_{I\bar{J}}$, $\mathcal{N}_{IJ}$ and N_{IJ} depend only on ratios of the fields by virtue of the homogeneity of the function $F(X)$. Therefore it is convenient to introduce n independent fields through the ratios $z^A \equiv X^A/X^0$ $(A=1,\ldots,n)$. These coordinates are sometimes called *special* coordinates [6, 23]. Including $z^0=1$ we can straightforwardly replace all fields X^I in the above equations by z^I.

The Lagrangian of these n scalar fields z^A takes the form of a non-linear sigma model corresponding to a Kähler manifold. Its Kähler potential is

$$K(z,\bar{z}) = \ln Y(z,\bar{z}) = \ln N_{IJ}\, z^I\, \bar{z}^J, \tag{2.3}$$

so that the sigma model metric is given by

$$g_{A\bar{B}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^A\, \partial \bar{z}^B} = \frac{\mathcal{M}_{A\bar{B}}}{zN\bar{z}}\ , \tag{2.4}$$

where $zN\bar{z} \equiv z^I N_{IJ}\bar{z}^J$. The curvature tensor corresponding to this metric equals[2]

$$R^A{}_{BC}{}^D = -2\delta^A_{(B}\,\delta^D_{C)} - \frac{1}{(zN\bar{z})^2}\,Q_{BCE}\,\bar{Q}^{EAD}, \tag{2.5}$$

where

$$Q_{IJK} \equiv \tfrac{1}{4}X^0\,F_{IJK}, \quad \bar{Q}^{ABC} = g^{\bar{D}A}g^{\bar{E}B}g^{\bar{F}C}\bar{Q}_{DEF}. \tag{2.6}$$

The scalar fields z^A are constrained to a domain defined by the requirement that the kinetic terms be positive. This is the case when $Y(z,\bar{z})$ is positive and $\mathcal{M}_{A\bar{B}}$ is negative definite (as shown in [15] this ensures that the matrix $\mathcal{N}_{IJ}+\bar{\mathcal{N}}_{IJ}$ is negative definite). The *special* Kähler manifolds are thus fully specified in terms of a homogeneous holomorphic functions of second degree. One may also describe special geometry on the basis of (2.5) in a coordinate-independent way [6, 23].

It is known that terms in $F(X)$ that are quadratic polynomials in X with *imaginary* coefficients, do not contribute to the action. In addition, it is possible that two different functions $F(X)$ and $\tilde{F}(\tilde{X})$, where the new fields $\tilde{X}^I$ can be expressed in terms of the old ones, give rise to the same theory in the sense that their equations of motion are equivalent [3] (at least for abelian vector fields). The reparametrizations that relate

[2] For the precise definitions of the Kähler curvature and connections, see the beginning of section 4.

X and $\tilde{X}$ are called *symplectic*, because the $(2n+2)$-component vectors $(X^I, -\frac{1}{2}iF_I)$ and $(\tilde{X}^I, -\frac{1}{2}i\tilde{F}_I)$ are related by constant $Sp(2n+2,\mathbb{R})$ matrices $\mathcal{O}$. Here $\tilde{F}_I$ is the derivative with respect to $\tilde{X}^I$ of the new function $\tilde{F}(\tilde{X})$, which can be determined from the symplectic matrix and the original function $F(X)$. The symplectic reparametrizations are discussed in more detail in appendix C. There we will also exhibit how the other fields of the vector supermultiplets transform under the symplectic reparametrizations.[3]

When the function $F(X)$ does not change under a symplectic reparametrization, i.e., when

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) \ , \tag{2.7}$$

then one has a so-called *duality invariance*, which gives rise to an isometry of the non-linear sigma model. For infinitesimal transformations, where the $Sp(2n+2,\mathbb{R})$ matrix is parametrized by

$$\mathcal{O} = \mathbb{1} + \begin{pmatrix} B & -D \\ C & -B^{\mathrm{T}} \end{pmatrix}, \tag{2.8}$$

with $B^I{}_J$, C_{IJ} and D^{IJ} real constant $(n+1)\times(n+1)$ matrices and C and D symmetric, the consistency of the symplectic transformation and the identification (2.7) requires the condition [5]

$$iC_{IJ}\, X^I X^J - B^I{}_J\, F_I X^J - \tfrac{1}{4} i D^{IJ}\, F_I F_J = 0 \ . \tag{2.9}$$

For the scalar sector this produces invariances of the action under

$$\delta X^I = B^I_J\, X^J + \tfrac{1}{2} i D^{IJ}\, F_J \ . \tag{2.10}$$

Observe that (2.7) does not imply that the function $F(X)$ is invariant under (2.10); instead one finds

$$\delta F(X) \equiv F(\tilde{X}) - F(X) = i\left(C_{IJ}\, X^I X^J + \tfrac{1}{4} D^{IJ}\, F_I F_J\right) \ . \tag{2.11}$$

For further details of these transformations we refer to appendix C. Finally, we should stress, that one cannot exclude the possibility that the sigma model possesses more isometries than just those corresponding to the above duality transformations; these additional isometries would then be broken by the interaction with the vector fields.

This completes our introduction to special Kähler manifolds characterized by homogeneous holomorphic functions $F(X)$, modulo quadratic polynomials with imaginary coefficients and $Sp(2n+2,\mathbb{R})$ reparametrizations. Let us now briefly discuss the class of theories corresponding to the functions

$$F(X) = i d_{ABC} \frac{X^A X^B X^C}{X^0}, \tag{2.12}$$

with d_{ABC} a symmetric real tensor. As we shall discuss shortly, these theories follow from five-dimensional supergravity by dimensional reduction, so that the sigma models corresponding to (2.12) constitute the image of the **r** map. Supergravity theories based on these functions can lead to flat potentials, as was shown in [15]. Furthermore they appear in the low-energy sector of certain superstring compactifications on (2,2) superconformal theories [3] and exhibit Peccei-Quinn-like symmetries as is appropriate for certain

[3]For Calabi-Yau manifolds these symplectic transformations are naturally induced on the periods by changes in the corresponding cohomology basis (see, e.g. [7, 8]).

(classical) superstring compactifications. There are arguments indicating that this class of functions comprises all the homogeneous Kähler spaces that can be coupled to $N=2$ supergravity [16, 17].

The functions (2.12) lead to the following Kähler metric, which depends only on the imaginary parts of the coordinates z^A ,

$$g_{A\bar{B}} = \frac{\mathcal{M}_{A\bar{B}}}{zN\bar{z}} = 6\,\frac{(d\,x)_{AB}}{(d\,xxx)} - 9\,\frac{(d\,xx)_A\,(d\,xx)_B}{(d\,xxx)^2}, \tag{2.13}$$

where $x^A \equiv i(z^A - \bar{z}^A)$, $(d\,x)_{AB} = d_{ABC}\,x^C$, $(d\,xx)_A = d_{ABC}\,x^Bx^C$ and $(d\,xxx) = d_{ABC}\,x^Ax^Bx^C$. Furthermore we have

$$Q_{ABC} = \tfrac{3}{2}i\,d_{ABC}. \tag{2.14}$$

The curvature corresponding to (2.13) follows from (2.5),

$$R^A{}_{BC}{}^D = -2\delta^A_{(B}\,\delta^D_{C)} + \tfrac{4}{3}C^{ADE}\,d_{BCE}\ , \tag{2.15}$$

where C^{ABC} is defined by

$$C^{ABC} = -\tfrac{9}{8}i\,(zN\bar{z})^{-2}\,\bar{Q}^{ABC} = -27g^{A\bar{D}}g^{B\bar{E}}g^{C\bar{F}}d_{DEF}(d\,xxx)^{-2}. \tag{2.16}$$

The theory based on (2.12) is always invariant under duality invariances. Imposing the condition (2.9) we find the following form for the matrices B^I_J, C_{IJ} and D^{IJ} (the first row and column refer to the $I=0$ component) [15],

$$B^I_J = \begin{pmatrix} \beta & a_B \\ b^A & \tilde{B}^A_B + \frac{1}{3}\beta\,\delta^A_B \end{pmatrix},\quad C_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 3d_{ABC}\,b^C \end{pmatrix},\quad D^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & D^{AB} \end{pmatrix}. \tag{2.17}$$

The isometries associated with the parameters β and b^A are always present; those depending on $\tilde{B}^A_B$ correspond to the symmetries of d_{ABC}, i.e.,

$$\tilde{B}^D_{(A}\,d_{BC)D} = 0. \tag{2.18}$$

One can prove that the matrix D^{AB} is proportional to the parameters a_A,

$$D^{AB} = -\tfrac{4}{9}C^{ABC}\,a_C\ , \tag{2.19}$$

while these parameters are subject to the condition

$$a_G\,E^G_{ABCD} = 0\ , \tag{2.20}$$

with the tensor E^G_{ABCD} defined by

$$E^G_{ABCD} = C^{EFG}\,d_{E(AB}\,d_{CD)F} - \delta^G_{(A}\,d_{BCD)}\ . \tag{2.21}$$

Obviously the matrix D^{AB} and the parameters a^A must be constant, so that $C^{ABC}a_C$ (and therefore $R^A{}_{BC}{}^D a_D$) is constant. In section 4.2 we will show that (2.20) is a necessary and sufficient condition for this to be the case.

In terms of the special coordinates the above isometries imply the following transformation

$$\delta z^A = b^A - \tfrac{2}{3}\beta\, z^A + \tilde{B}^A_B\, z^B + \tfrac{1}{2}(R^A{}_{BC}{}^D\, a_D)\, z^B z^C. \tag{2.22}$$

Clearly the transformations characterized by $\tilde{B}^A_B$ and β form two commuting subgroups of the full group of duality transformations. The root lattice of the algebra corresponding to all duality transformations consists of the root lattice of the generators corresponding to $\tilde{B}^A_B$ extended with one dimension associated with the eigenvalues of the roots under the β-symmetry. This leads to a characteristic lattice such as shown in Fig. 1 for an $n = 3$ special Kähler space based on the function $F(X) = 3i\,(X^2/X^0)\,(X^2X^1 - (X^3)^2)$. The

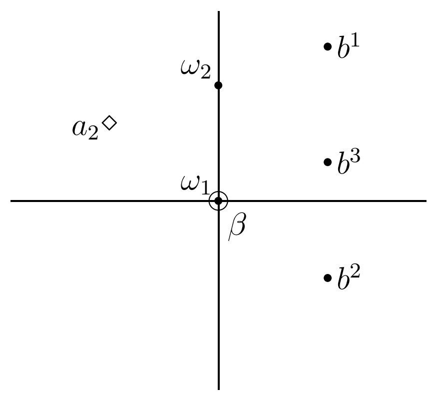


Figure 1: Root lattice corresponding to the isometries of an $n = 3$ special Kähler manifold discussed in the text.

subgroup associated with the matrices $\tilde{B}^A_B$ has two independent parameters denoted by ω_1 and ω_2. Their roots correspond to the solvable algebra of $SU(1,1)$, which is extended to a six-dimensional solvable algebra by the roots associated with the parameters β, b^1, b^2 and b^3. In this case there is one *hidden* symmetry associated with the parameter a_2 and indicated in the diagram by $\diamond$. For higher rank one obtains a similar lattice by projecting on a suitably chosen plane. This particular example is discussed in [14] and is the simplest case of the homogeneous non-symmetric spaces discussed in section 5 (it will be denoted by $L(-1,1)$).

2.2 Special real manifolds

The theories corresponding to the functions (2.12) can be obtained by dimensional reduction from Maxwell-Einstein supergravity theories in $d = 5$ space-time dimensions [13]. This theory contains $n-1$ real scalar fields and n vector fields (one of them corresponding to the graviphoton). The Lagrangian corresponding to these fields reads

$$\begin{aligned} e^{-1}\mathcal{L} \;&=\; -\tfrac{1}{2}R - \tfrac{3}{2}\, d_{ABC}\, h^A\, \partial_\mu h^B\, \partial^\mu h^C \\ &\quad +\tfrac{1}{2}\, \left(6(d\,h)_{AB} - 9(d\,hh)_A\,(d\,hh)_B\right) F^A_{\mu\nu}(A)\, F^{B\,\mu\nu}(A) \\ &\quad + e^{-1} i\epsilon^{\mu\nu\rho\sigma\lambda} d_{ABC}\, F^A_{\mu\nu}(A)\, F^B_{\rho\sigma}(A)\, A^C_\lambda, \end{aligned} \tag{2.23}$$

where A_μ^A and $F_{\mu\nu}^A(A)$ denote the gauge fields and their corresponding abelian field strengths and the scalar fields h^A are subject to the condition

$$d_{ABC}\, h^A h^B h^C = 1. \tag{2.24}$$

The scalar fields must again be restricted to a domain so that all kinetic terms in (2.23) have the required signature.

After dimensional reduction, the Lagrangian (2.23) becomes equal to (2.1); the imaginary part of the four-dimensional scalar fields z^A originate from the components of the gauge fields A_μ^A in the fifth dimension, while their real part corresponds to the $n-1$ independent fields h^A and the component g_{55} of the metric. The $n+1$ gauge fields in four dimensions are related to the n gauge fields in five dimensions and the off-diagonal components $g_{\mu 5}$ of the metric in five dimensions.

The Lagrangian (2.23) is manifestly invariant under linear transformations of the fields

$$h^A \to \tilde{B}^A_{\ B}\, h^B, \qquad A_\mu^A \to \tilde{B}^A_{\ B}\, A_\mu^B, \tag{2.25}$$

provided that the matrices $\tilde{B}$ leave the tensor d_{ABC} invariant (cf. 2.18). After reduction to four space-time dimensions a number of extra symmetries emerges, which find their origin in the five-dimensional theory. First of all, the extra vector field emerging from the five-dimensional metric has a corresponding gauge invariance related to reparametrizations of the extra fifth coordinate by functions that depend only on the four space-time coordinates. Then there are special gauge transformations of the n vector fields with gauge functions that depend exclusively and linearly on the fifth coordinate. Under these transformations the fifth component of each of the gauge fields transforms with a constant translation, whereas the remaining four-dimensional gauge fields transform linearly into the gauge field originating from the five-dimensional metric (this last transformation arises because of certain field redefinitions that must be performed on the vector fields for reasons of $d=4$ general covariance). As only the field equations are gauge invariant (the Lagrangian is not gauge invariant in view of the last term in (2.23)), it comes as no surprise that these invariances will manifest themselves as duality invariances of the four-dimensional field equations corresponding to the Lagrangian (2.1). Indeed, these transformations are the ones associated with the parameters b^A.

The same phenomenon takes place for scale transformations of the fifth coordinate, which do not leave the Lagrangian invariant either. In the four-dimensional reduction these transformations correspond to the duality invariances associated with the parameter β. However, this relationship is somewhat less direct, as it has to be defined such that the properly defined four-dimensional metric remains invariant under this transformation. The latter may not be so obvious in view of the standard Weyl rescaling that is required in order to obtain the standard Einstein-Hilbert action after dimensional reduction. We should emphasize that none of the extra symmetries emerging from the dimensional reduction act on the original five-dimensional scalar fields. Of course, one may have performed field redefinitions after the reduction that obscure this fact.

What remains are the possible extra duality invariances of the four-dimensional theory that are proportional to the parameters a_A. These transformations have no obvious five-dimensional origin and their presence depends entirely on the non-trivial restriction (2.20). We shall denote such symmetries as *hidden* symmetries, to distinguish them from those whose existence can be inferred on more general grounds, as explained above. However, the roots corresponding to these extra symmetries are all located on the left half-plane in Fig. 1. To be more precise, we can decompose the symmetry algebra $\mathcal{W}$ corresponding to all the roots into eigenspaces of the generators associated with the β symmetry. We then find the following decomposition

$$\mathcal{W} = \mathcal{W}_{-2/3} + \mathcal{W}_0 + \mathcal{W}_{2/3}, \tag{2.26}$$

where the subscript denotes the eigenvalue with respect to the β symmetry. As it turns out, $\mathcal{W}_0$ denotes the subalgebra associated with the parameters $\tilde{B}^A_B$ and β. $\mathcal{W}_{2/3}$ contains the generators corresponding to the parameters b^A and all possible generators corresponding to the hidden symmetries with parameters a_A belong to $\mathcal{W}_{-2/3}$. Observe that the dimension of $\mathcal{W}_{-2/3}$ is at most equal to n, whereas the dimension of $\mathcal{W}_{2/3}$ is always equal to n. Unless we have maximal symmetry (i.e., unless there are n independent symmetries associated with the parameters a_A) the isometry group of the corresponding Kähler space is not semisimple. The maximal number of isometries exists when the curvature and the tensor C^{ABC} are constant (in special coordinates), or equivalently, when the tensor E^G_{ABCD} defined in (2.21) vanishes. In that case it is known that the corresponding Kähler space is symmetric [15].

2.3 Special quaternionic manifolds

We now return to the general four-dimensional Lagrangian (2.1) based on a general function $F(X)$ and consider its reduction to three space-time dimensions. In this case an extra feature is present, because the standard (abelian) gauge field Lagrangian in three dimensions can be converted to a scalar field Lagrangian by means of a duality transformation. Only derivatives of this scalar field appear, so that it has a corresponding invariance under constant shifts. Each four-dimensional gauge field thus gives rise to two scalar fields with two corresponding isometries. One is its component in the fourth dimension and the other is the scalar field that results from the conversion of the three-dimensional gauge field. The $n+1$ four-dimensional vector fields A^I_μ thus give rise to $2n+2$ scalar fields, which will be denoted by A^I and B_I. The corresponding invariances have parameters α^I and β_I. The same conversion can be used for the vector field that emerges from the higher-dimensional metric, so that the four-dimensional metric gives rise to a three-dimensional metric and two scalar fields. These scalar fields are denoted by ϕ and σ, and they also lead to two invariances, one related to the scale transformation of the extra coordinate with parameter ϵ^0 and another one corresponding to the converted three-dimensional vector field with parameter ϵ^+. Altogether, the Lagrangian (2.1) thus gives rise to $4(n+1)$ scalar fields, coupled to gravity with $2n+4$ additional invariances. As there are no vector fields anymore, the corresponding transformations must constitute an invariance of the Lagrangian. As this Lagrangian is still locally supersymmetric the scalar fields must define a quaternionic non-linear sigma model [24]. The corresponding spaces are called *special* quaternionic spaces and obviously depend on a homogeneous holomorphic function of second degree. They constitute the image of the **c** map, as was explained in section 1. As all quaternionic spaces, they are irreducible Einstein spaces [25]

The Lagrangian for this quaternionic sigma model was determined in [26], where the quaternionic structure was verified, and in [19]. It reads

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& -\tfrac{1}{2}R + (zN\bar{z})^{-1}\mathcal{M}_{A\overline{B}}\,\partial_\mu z^A\,\partial^\mu \bar{z}^B \\
& +\tfrac{1}{4}\phi^{-1}(\mathcal{N}+\overline{\mathcal{N}})_{IJ}\,W_\mu^I\,\overline{W}^{J\mu} \\
& -\tfrac{1}{4}\phi^{-2}\Big\{(\partial_\mu\sigma - \tfrac{1}{2}A^I\overleftrightarrow{\partial}_\mu B_I)^2 + (\partial_\mu\phi)^2\Big\},
\end{aligned}
\tag{2.27}
$$

where

$$
W_\mu^I \;=\; (\mathcal{N}+\overline{\mathcal{N}})^{-1\,IJ}\Big[2\overline{\mathcal{N}}_{JK}\partial_\mu A^K - i\partial_\mu B_J\Big] \tag{2.28}
$$

is directly related to the field strengths of the original vector fields. Again the scalar fields are restricted to a domain $\mathcal{D}$ in which the kinetic terms have the required signature. From the fact that the field strengths $(W_\mu^I, 2i\mathcal{N}_{IJ}W_\mu^J)$ transform under $Sp(2n+2,\mathrm{I\!R})$ just as $(X^I, -\frac{1}{2}iF_I)$, one can show that the scalar fields A^I and B_I transform linearly in the same $(2n+2)$-dimensional representation. The fields ϕ and σ are invariant under the duality transformations. On the other hand, as mentioned previously, the original scalar fields z^A are inert under the $2n+4$ extra symmetries associated with the parameters α^I, β_I, ϵ^+ and ϵ^0. The full symmetry variations of the fields will be given in (3.4).

Let us again consider the root lattice corresponding to all these symmetries. It consists of the root lattice of the duality invariance extended by one dimension associated with the eigenvalue of the generator $\underline{\epsilon}_0$ (from now on we use the notation where generators are denoted by their corresponding parameter, underlined with their index raised or lowered). This leads to a root lattice such as shown in Fig. 2 (for the moment ignore the generators

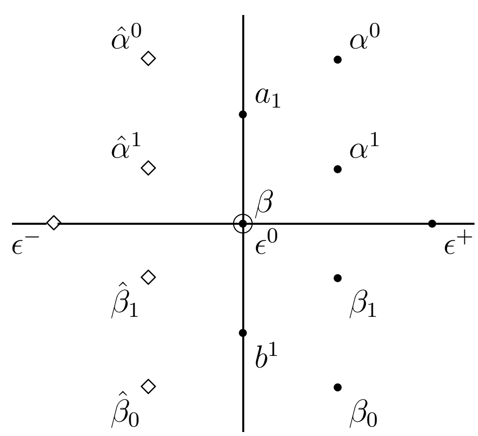


Figure 2: Root lattice corresponding to the isometries of an $n=1$ special quaternionic manifold discussed in the text.

indicated by $\diamond$), where we have exhibited the case where the duality invariances constitute a group of rank 1 (namely $SU(1,1)$). The space has $n=1$ and is based on $F(X) = i(X^1)^3/X^0$. The roots corresponding to these duality isometries are extended with the roots belonging to the extra transformations that emerge from the reduction to three dimensions. Observe that there are no duality invariances associated with the matrices $\tilde{B}^A_B$ in this case. Additional *hidden* symmetries are located on the left half-plane. In the case at hand, there are five such symmetries indicated by $\diamond$, which extend this diagram to the root lattice of $G_{2(+2)}$. Its solvable subalgebra consists of the solvable subalgebra of $SU(1,1)$, associated with the generators $\underline{\beta}$ and $\underline{b}_1$, extended by the generators $\underline{\epsilon}_0$, $\underline{\epsilon}_+$, $\underline{\alpha}_I$ and $\underline{\beta}^I$ of the six extra symmetries. In the general case we obtain a similar diagram after projecting all the roots on a suitably chosen plane.

The algebra $\mathcal{V}$ corresponding to the roots of the Kähler space isometries and the extra symmetries, which is obviously non-semisimple, can generally be decomposed into eigenspaces of the generator $\underline{\epsilon}_0$ in the adjoint representation. One finds

$$\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{1/2} + \mathcal{V}_1, \tag{2.29}$$

where $\mathcal{V}_a$ denotes the eigenspace with eigenvalue a. It turns out that $\mathcal{V}_0$ contains the generators of the duality invariance, previously denoted by $\mathcal{W}$, supplemented by the generator $\underline{\epsilon}_0$, $\mathcal{V}_{1/2}$ contains the $2n{+}2$ generators $\underline{\alpha}_I$ and $\underline{\beta}^I$ and $\mathcal{V}_1$ consists of the generator $\underline{\epsilon}_+$. As we shall discuss shortly, there exist no other symmetries with non-negative eigenvalues of $\underline{\epsilon}_0$, so that the root decomposition (2.29) is complete in that respect [19].

The decomposition (2.29) was also encountered in the classification of *normal* quaternionic spaces given by Alekseevskiĭ [21]. Normal quaternionic spaces are quaternionic spaces that admit a transitive completely solvable group of motions. It was conjectured in [21] that the homogeneous quaternionic spaces consist of compact symmetric quaternionic and normal quaternionic spaces. The algebra corresponding to the group of solvable motions in the latter case exhibits the same decomposition as in (2.29), although the spaces that emerge under the **c** map are not necessarily normal quaternionic. According to Alekseevskiĭ there are two different types of normal quaternionic spaces characterized by their so-called canonical quaternionic subalgebra. The first type with subalgebra C^1_1 turns out to correspond to the quaternionic projective spaces $Sp(m,1)/(Sp(m)\otimes Sp(1))$. Their *solvable* algebra $\mathcal{V}^{\rm s}$ decomposes as in (2.29), where $\mathcal{V}^{\rm s}_0$ contains only the generator $\underline{\epsilon}_0$, while $\mathcal{V}^{\rm s}_{1/2}$ and $\mathcal{V}^{\rm s}_1$ have dimension $4n$ and 3, respectively. As mentioned above, $\mathcal{V}_1$ is always one-dimensional, so that the quaternionic projective spaces are *not* in the image of the **c** map.

The second type has a canonical subalgebra A^1_1 and the structure of the solvable algebra $\mathcal{V}^{\rm s}$ is as follows: $\mathcal{V}^{\rm s}_0$ is a direct sum of the generator $\underline{\epsilon}_0$ and a normal Kähler algebra $\mathcal{W}^{\rm s}$ of dimension $2n$, $\mathcal{V}^{\rm s}_{1/2}$ has dimension $2n+2$ and $\mathcal{V}^{\rm s}_1$ has dimension 1. In order to be quaternionic, the representation of $\mathcal{W}^{\rm s}$ induced by the adjoint representation of $\mathcal{V}^{\rm s}$ on $\mathcal{V}^{\rm s}_{1/2}$ must generate a solvable subgroup of $Sp(2n+2,\mathbb{R})$. Hence the special quaternionic spaces have the same root structure (cf. 2.29), except that the algebra $\mathcal{W}$ is not necessarily solvable. For a normal Kähler space with duality transformations acting transitively on the manifold, $\mathcal{W}$ has a $2n$-dimensional solvable subalgebra, so that the **c** map yields a normal quaternionic space. Conversely each normal quaternionic space with canonical subalgebra A^1_1 defines the basic ingredients of a special normal Kähler space, encoded in its solvable transitive group of duality transformations. Alekseevskiĭ's analysis thus indicates that all these spaces are in the image of the **c** map. To establish the existence of the corresponding four-dimensional supergravity theory, one must prove that a holomorphic function $F(X)$ exists that allows for these duality transformations (i.e., that satisfies (2.9)). This program was carried out by Cecotti [16], who explicitly constructed the function $F(X)$ corresponding to each of the normal quaternionic spaces with canonical subalgebra A_1^1. With the exception of the so-called minimal coupling, where $F(X)$ is a quadratic polynomial, all functions $F(X)$ can be brought into the form (2.12). Recently it turned out that this conclusion also holds for certain spaces that were missing in Alekseevskiĭ's classification (cf. [17]).

We already alluded to possible additional (hidden) symmetries for the special quaternionic manifolds, in analogy with the symmetries associated with the parameters a_A in the Kähler case (cf. Fig. 1). This question was analyzed in [19] and it was found that the hidden symmetries are always associated to roots with eigenvalue -1 or $-1/2$ with respect to the generator $\underline{\epsilon}_0$. The root lattice corresponding to all symmetries of the Lagrangian (2.27) thus generally decomposes according to

$$\mathcal{V} = \mathcal{V}_{-1} + \mathcal{V}_{-1/2} + \mathcal{V}_0 + \mathcal{V}_{1/2} + \mathcal{V}_1. \tag{2.30}$$

As stressed previously, the roots with non-negative eigenvalues were already completely specified in (2.29). The dimensions of $\mathcal{V}_{-1}$ and $\mathcal{V}_{-1/2}$ are smaller or equal to 1 and $2n+2$, respectively. The parameters associated with the new symmetries are denoted by ϵ^-, corresponding to $\mathcal{V}_{-1}$, and $\hat{\alpha}^I$ and $\hat{\beta}_I$, corresponding to the generators in $\mathcal{V}_{-1/2}$.

The full symmetry structure of the quaternionic manifold is encoded in a single real function h,

$$\begin{aligned} h(X,\bar X,A,B) \;=\; & -\tfrac{1}{16}\Big\{(\mathcal{B}_I\bar{\mathcal{B}}^I)^2 - \tfrac{1}{6}\Big[(F_{IJK}\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J\bar{\mathcal{B}}^K)(X^L\mathcal{B}_L) + h.c.\Big] \\ & +\tfrac{1}{16}(XN\bar X)\bar{\mathcal{B}}^I\bar{\mathcal{B}}^J F_{IJK} N^{-1\,KL}\bar F_{LMN}\mathcal{B}^M\mathcal{B}^N\Big\}\,, \end{aligned} \tag{2.31}$$

where

$$\mathcal{B}_I = B_I + \tfrac{1}{2} i F_{IJ} A^J \equiv N_{IJ}\mathcal{B}^J. \tag{2.32}$$

The function h is a real quartic polynomial in the fields A^I and B_I. Furthermore it is homogeneous of zeroth degree in X and $\bar X$ separately, as well as invariant under duality transformations (to see this, use the results of appendix C).

The maximal number of symmetries exists if and only if the function h is independent of X^I and $\bar X^I$. In that case the dimensions of $\mathcal{V}_{-1}$ and $\mathcal{V}_{-1/2}$ are equal to 1 and $2n+2$, respectively, and the isometry group is semisimple. The derivative of h with respect to X is proportional to a fully symmetric real tensor C_{IJKL} [19],

$$C_{IJKL} \equiv \tfrac{1}{4} F_{IJKL}(XN\bar X) + (N\bar X)_{(I}F_{JKL)} - \tfrac{3}{16} F_{M(IJ}N^{-1\,MN}F_{KL)N}(XN\bar X), \tag{2.33}$$

which satisfies the relations

$$X^I C_{IJKL} = 0, \tag{2.34}$$

$$\bar X^M \frac{\partial}{\partial \bar X^M} C_{IJKL} = -X^M\frac{\partial}{\partial X^M} C_{IJKL} = C_{IJKL}. \tag{2.35}$$

The vanishing of this tensor is a necessary and sufficient condition for the Kähler manifold (as well as the associated quaternionic manifold) to be symmetric, because of the relation [27]

$$R^A{}_{BC}{}^D{}_{;E} = -(zN\bar z)^{-2}\,\bar Q^{ADF}\,\mathcal{C}_{FBCE}\ , \tag{2.36}$$

where

$$\mathcal{C}_{ABCD} \equiv \frac{(X^0)^2}{XN\bar X} C_{ABCD} \tag{2.37}$$

is a tensor that depends on z and $\bar{z}$. In view of (2.34) $\mathcal{C}_{ABCD}$ contains all independent components of C_{IJKL}.

When h is not X-independent, hidden symmetries belonging to $\mathcal{V}_{-1/2}$ can still exist provided that there are linear combinations of first-order derivatives of h with respect to A^I and/or B_I that are independent of X^I. Hence, the situation can be summarized as follows. The hidden symmetries correspond to the independent parameters ϵ_-, $\hat{\alpha}^I$ and $\hat{\beta}_I$ for which

$$\mathcal{D}h = \left(\epsilon^- + \hat{\alpha}^I \frac{\partial}{\partial A^I} + \hat{\beta}_I \frac{\partial}{\partial B_I}\right) h \tag{2.38}$$

is independent of X^I and $\bar{X}^I$. This condition is equivalent to the following conditions on the tensor C_{IJKL},

$$C_{IJKL}\bar{\xi}^L = 0 \ , \tag{2.39}$$

$$\xi^M \frac{\partial}{\partial \bar{X}^M} C_{IJKL} = 0 \ , \tag{2.40}$$

where ξ^I is a function of the parameters $\hat{\alpha}^I$ and $\hat{\beta}_I$:

$$\xi_I = \hat{\beta}_I + \tfrac{1}{2} i F_{IJ}\hat{\alpha}^J \equiv N_{IJ}\xi^I \ . \tag{2.41}$$

These $2n+2$ parameters represent only independent solutions of the above equations when the space is symmetric and the symmetry associated with the generator $\underline{\epsilon}_-$ is realized.

The conditions (2.39) and (2.40) can be written exclusively in terms of $\mathcal{C}_{ABCD}$. Using (2.37) one rewrites (2.40) as

$$\left(X^J\xi_J + (XN\bar{X})\xi^J \frac{\partial}{\partial \bar{X}^J}\right) \mathcal{C}_{ABCD} = 0 \ . \tag{2.42}$$

Using (2.34) and (2.35) and the fact that $\mathcal{C}_{ABCD}$ depends only on z and $\bar{z}$, one can write (2.39) and (2.42) as

$$\mathcal{C}_{ABCD}\bar{\hat{\xi}}^D = 0 \ , \tag{2.43}$$

$$\left(z^J\xi_J + (zN\bar{z})\,\hat{\xi}^E \frac{\partial}{\partial \bar{z}^E}\right) \mathcal{C}_{ABCD} = 0 \ , \tag{2.44}$$

where $\hat{\xi}^A = \xi^A - \bar{z}^A\,\xi^0$. A more convenient expression follows from invoking (B.5),

$$\hat{\xi}^A = (zN\bar{z})^{-1} g^{\bar{A}B}\xi_B - \Delta^{-1}\left(n^{-1}N\bar{z}\right)^A \xi_I z^I \ , \tag{2.45}$$

where $\Delta^{-1} = \left(N^{-1}\right)^{00}$ (alternative expressions for Δ are given in (B.3)). We shall return to the above conditions in section 4, where we apply them to the manifolds described by the functions (2.12).

3 The symmetry algebra of special Kähler and quaternionic spaces

In this section we discuss the algebras of the isometries of generic special Kähler and quaternionic spaces and their consequences. In subsection 2.1 we have exhibited the isometries of a generic special Kähler space that coincide with the generalized duality transformations of the equations of motion corresponding to the Lagrangian (2.1). The infinitesimal transformations act on the scalars according to (2.10); the matrices C_{IJ}, $B^I{}_J$ and D^{IJ}, which constitute an infinitesimal $Sp(2n+2,\mathbb{R})$ transformation, are solutions of the consistency equation (2.9). We parametrize these solutions in terms of a number of independent transformation parameters ω^i. Using the notation that the generators corresponding to the parameters ω^i are denoted by $\underline{\omega}_i$, an arbitrary infinitesimal duality transformation is generated by

$$\delta(\omega) = \omega^i\,\underline{\omega}_i\ . \tag{3.1}$$

As the algebra of these transformations should close, we have

$$[\delta(\omega_1), \delta(\omega_2)] = \delta(\omega_3)\ . \tag{3.2}$$

The explicit form of ω_3 in terms of ω_1 and ω_2 can be obtained from commuting the corresponding $Sp(2n+2,\mathbb{R})$ matrices (2.8)[4]. We remind the reader that the manifold may have additional isometries *not* corresponding to duality transformations. Indeed, it was demonstrated in [14] that special real manifolds exist with a larger invariance group than that of the full supergravity Lagrangian. For the special Kähler manifolds we know of no examples where such a situation is realized, and for special Kähler spaces based on the functions (2.12) it was proven in [14] that the sigma model isometries are contained in the duality transformations. Such a proof is lacking for the generic manifolds, but in this section we shall ignore possible isometries of the Kähler space that are not contained in the group of duality transformations.

The isometry group enlarges considerably under the **c** map, which extends the special Kähler space to a special quaternionic space. The quaternionic isometries exhibit a systematic structure as expressed by the root decomposition (2.30). A typical example of this decomposition was shown in the root lattice Fig. 2. In the notation introduced in subsection 2.3, the infinitesimal isometries for special quaternionic manifolds are generated by

$$\delta = \epsilon^0\,\underline{\epsilon}_0 + \epsilon^+\,\underline{\epsilon}_+ + \epsilon^-\,\underline{\epsilon}_- + \alpha^I\,\underline{\alpha}_I + \beta_I\,\underline{\beta}^I + \hat\alpha^I\,\underline{\hat\alpha}_I + \hat\beta_I\,\underline{\hat\beta}^I + \omega^i\,\underline{\omega}_i\ . \tag{3.3}$$

We remind the reader that the "extra" symmetries parametrized by ϵ^+, ϵ^0, α^I and β_I are realized for any special quaternionic manifold. The generators of the "hidden" symmetries, belonging to the subspaces $\mathcal{V}_{-1}$ and $\mathcal{V}_{-1/2}$ of the root lattice are denoted by $\underline{\epsilon}_0$ and by $\underline{\hat\alpha}_I$ and $\underline{\hat\beta}^I$, respectively, exist whenever the appropriate conditions are satisfied. As we discussed in subsection 2.3, the symmetry in $\mathcal{V}_{-1}$, parametrized by ϵ_-, is only realized for symmetric spaces, characterized by the vanishing of the tensor C_{IJKL} given in (2.33). If this is not the case other "hidden" symmetries belonging to $\mathcal{V}_{-1/2}$ can be realized depending on whether the conditions (2.39) and (2.40) are satisfied.

Let us now turn to the infinitesimal transformations corresponding to (3.3) and acting on the coordinates ϕ, σ, A^I, B_I and X^I of the quaternionic manifold (it is more convenient to use X^I rather than z^A),

$$\delta\phi \;=\; \phi\left(-\epsilon^0 + 2\sigma\epsilon^- + \hat\alpha^I B_I - \hat\beta_I A^I\right)\ ,$$

[4]Note that when the infinitesimal transformation $\delta(\omega)$ is described by a matrix $\mathcal{B}(\omega)$ (as in (2.8)), then the commutator corresponding to (3.2) reads $\mathcal{B}(\omega_3) = [\mathcal{B}(\omega_2), \mathcal{B}(\omega_1)]$, with ω_1 and ω_2 in opposite order.

$$
\begin{aligned}
\delta\sigma &= \epsilon^+ + \tfrac{1}{2}\left(\alpha^I B_I - \beta_I A^I\right) + (\sigma^2-\phi^2)\epsilon^- + \sigma\left(\tfrac{1}{2}\hat\alpha^I B_I - \tfrac{1}{2}\hat\beta_I A^I - \epsilon^0\right) + \mathcal{D}h \;, \\
\delta A^I &= \alpha^I + \sigma\hat\alpha^I + B^I{}_J(\omega)\,A^J - D^{IJ}(\omega)\,B_J \\
&\quad + \left(\epsilon^-\sigma + \tfrac{1}{2}\hat\alpha^J B_J - \tfrac{1}{2}\hat\beta_J A^J - \tfrac{1}{2}\epsilon^0\right)A^I - \partial^I\mathcal{D}\left(h + \tfrac{1}{2}\phi\mathcal{Z}_2\right)\;, \\
\delta B_I &= \beta_I + \sigma\hat\beta_I + C_{IJ}(\omega)\,A^J - B^J{}_I(\omega)\,B_J \\
&\quad + \left(\epsilon^-\sigma + \tfrac{1}{2}\hat\alpha^J B_J - \tfrac{1}{2}\hat\beta_J A^J - \tfrac{1}{2}\epsilon^0\right)B_I + \partial_I\mathcal{D}\left(h + \tfrac{1}{2}\phi\mathcal{Z}_2\right)\;, \\
\delta X^I &= B^I{}_J(\omega)\,X^J + \tfrac{1}{2}iD^{IJ}(\omega)\,F_J \\
&\quad + \mathcal{D}\left(-\tfrac{1}{2}i\bar{\mathcal{B}}^I\,(X^J\mathcal{B}_J) + \tfrac{1}{16}i(N^{-1})^{IJ}\mathcal{B}^K\bar F_{JKL}\mathcal{B}^L\,(XN\bar X)\right)\;,
\end{aligned}
\tag{3.4}
$$

where $h(X,\bar X,A,B)$ is the function given in (2.31), $\mathcal{D}$ was defined in (2.38), ∂_I and ∂^I denote the derivatives with respect to A^I and B_I, respectively, and

$$
\mathcal{Z}_2 \equiv \mathcal{B}_I\bar{\mathcal{B}}^I - 2\frac{(X^I\mathcal{B}_I)(\bar X^J\bar{\mathcal{B}}_J)}{XN\bar X}\;. \tag{3.5}
$$

We remind the reader that the condition for the existence of "hidden" symmetries is that $\mathcal{D}h$ does not depend on X^I or $\bar X^I$.

The non-zero commutation relations of the above transformations are as follows. First we list those that do not involve the duality transformations,

$$
\begin{array}{ll}
[\underline{\epsilon}_0,\underline{\epsilon}_\pm] = \pm\underline{\epsilon}_\pm\;, & [\underline{\epsilon}_-,\underline{\epsilon}_+] = 2\underline{\epsilon}_0\;, \\
[\underline{\epsilon}_0,\underline{\alpha}_I] = \tfrac{1}{2}\underline{\alpha}_I\;, & [\underline{\epsilon}_0,\underline{\hat\alpha}_I] = -\tfrac{1}{2}\underline{\hat\alpha}_I\;, \\
[\underline{\epsilon}_0,\underline{\beta}^I] = \tfrac{1}{2}\underline{\beta}^I\;, & [\underline{\epsilon}_0,\underline{\hat\beta}^I] = -\tfrac{1}{2}\underline{\hat\beta}^I\;, \\
[\underline{\epsilon}_-,\underline{\alpha}_I] = -\underline{\hat\alpha}_I\;, & [\underline{\epsilon}_+,\underline{\hat\alpha}_I] = \underline{\alpha}_I\;, \\
[\underline{\epsilon}_-,\underline{\beta}^I] = -\underline{\hat\beta}^I\;, & [\underline{\epsilon}_+,\underline{\hat\beta}^I] = \underline{\beta}^I\;, \\
[\underline{\alpha}_I,\underline{\beta}^J] = -\delta_I^J\underline{\epsilon}_+\;, & [\underline{\hat\alpha}_I,\underline{\hat\beta}^J] = -\delta_I^J\underline{\epsilon}_-\;.
\end{array}
\tag{3.6}
$$

Then we have

$$
\begin{aligned}
[\alpha^I\,\underline{\alpha}_I + \beta_I\,\underline{\beta}^I, \omega^i\,\underline{\omega}_i] &= \alpha'^I\,\underline{\alpha}_I + \beta'_I\,\underline{\beta}^I\;, \\
[\hat\alpha^I\,\underline{\hat\alpha}_I + \hat\beta_I\,\underline{\hat\beta}^I, \omega^i\,\underline{\omega}_i] &= \hat\alpha'^I\,\underline{\hat\alpha}_I + \hat\beta'_I\,\underline{\hat\beta}^I\;,
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{pmatrix} \alpha'^I \\ \beta'_J \end{pmatrix} = \begin{pmatrix} B^I{}_K(\omega) & -D^{IL}(\omega) \\ C_{JK}(\omega) & -B^L{}_J(\omega) \end{pmatrix} \begin{pmatrix} \alpha^K \\ \beta_L \end{pmatrix}\;, \tag{3.8}
$$

and likewise for $\hat\alpha'$ and $\hat\beta'$. Finally there are the commutators

$$
[\alpha^I\,\underline{\alpha}_I + \beta_I\,\underline{\beta}^I, \hat\alpha^I\,\underline{\hat\alpha}_I + \hat\beta_I\,\underline{\hat\beta}^I] = (\alpha^I\hat\beta_I - \hat\alpha^I\beta_I)\,\underline{\epsilon}_0 + \omega^i(\alpha,\beta,\hat\alpha,\hat\beta)\,\underline{\omega}_i\;. \tag{3.9}
$$

Explicit calculation based on (3.4) shows that the duality transformations corresponding to the parameters $\omega(\alpha,\beta,\hat\alpha,\hat\beta)$ correspond to the matrices

$$
\begin{aligned}
B^I{}_J(\alpha,\beta,\hat\alpha,\hat\beta) &= -\tfrac{1}{2}(\hat\alpha^I\beta_J + \alpha^I\hat\beta_J) - \partial^I\partial_J h''(\alpha,\beta,\hat\alpha,\hat\beta)\;, \\
C_{IJ}(\alpha,\beta,\hat\alpha,\hat\beta) &= -\hat\beta_{(I}\beta_{J)} + \partial_I\partial_J h''(\alpha,\beta,\hat\alpha,\hat\beta)\;, \\
D^{IJ}(\alpha,\beta,\hat\alpha,\hat\beta) &= -\hat\alpha^{(I}\alpha^{J)} + \partial^I\partial^J h''(\alpha,\beta,\hat\alpha,\hat\beta)\;,
\end{aligned}
\tag{3.10}
$$

where

$$h''(\alpha,\beta,\hat\alpha,\hat\beta) \equiv (\alpha\cdot\partial+\beta\cdot\partial)(\hat\alpha\cdot\partial+\hat\beta\cdot\partial)\, h(X,\bar X,A,B)\ . \tag{3.11}$$

From supersymmetry considerations, we know that the above algebra is quaternionic. Its structure can be visualized by a root diagram and its Cartan subalgebra consists of the Cartan subalgebra of the algebra corresponding to the duality transformations (the Kähler algebra) and the generator $\underline{\epsilon}_0$. As discussed in subsection 2.3 this leads to a typical diagram such as shown in Fig. 2, where the generators of the Kähler algebra are located on the vertical axis. It is clear that the rank of the quaternionic algebra is one unit higher than that of the corresponding Kähler algebra.

We already mentioned that when the symmetry associated with the generator $\underline{\epsilon}_-$ exists, then all $2n+3$ symmetries associated with $\mathcal{V}_{-1}$ and $\mathcal{V}_{-1/2}$ are realized. When we know of the existence of one hidden symmetry belonging to $\mathcal{V}_{-1/2}$, say $\hat\alpha_\star^I\,\underline{\hat\alpha}_I+\hat\beta_{\star I}\,\underline{\hat\beta}^I$, then important information regarding the symmetry structure follows from the algebra. We mention four such results:

- (i). Any other hidden symmetry belonging to $\mathcal{V}_{-1/2}$ and parametrized by $\hat\alpha^I\,\underline{\hat\alpha}_I+\hat\beta_I\,\underline{\hat\beta}^I$ should either satisfy $\hat\alpha\cdot\hat\beta_\star-\hat\beta\cdot\hat\alpha_\star=0$, or $\underline{\epsilon}_-$ is also a symmetry. The latter implies that all hidden symmetries in $\mathcal{V}_{-1}$ and $\mathcal{V}_{-1/2}$ are realized; in that case both the Kähler and the corresponding quaternionic space are symmetric.
- (ii). Also those hidden symmetries should exist that are related to $\hat\alpha_\star^I\,\underline{\hat\alpha}_I+\hat\beta_{\star I}\,\underline{\hat\beta}^I$ by the action of the duality transformations.
- (iii). There exists an expression for $(\hat\alpha_\star\cdot\partial+\hat\beta_\star\cdot\partial)h$, which is a cubic polynomial in A^I and B_I with constant coefficients that are severely restricted by the consistency equation (2.9).
- (iv). There exist non-trivial duality invariances.

The first two assertions follow directly from the last commutator in (3.6) and from (3.8). The last two assertions need further explanation. First consider the commutator of $\hat\alpha_\star^I\,\underline{\hat\alpha}_I+\hat\beta_{\star I}\,\underline{\hat\beta}^I$ with $\alpha^I\,\underline{\alpha}_I+\beta_I\,\underline{\beta}^I$, for general α and β. It leads to the duality transformations parametrized by the matrices (3.10) and implies that the third-order derivatives of $\left(\hat\alpha_\star\cdot\partial+\hat\beta_\star\cdot\partial\right)h$ with respect to A^I and B_I must be constant (as is indeed guaranteed by the defining condition for a hidden isometry!). Moreover the matrices (3.10) should satisfy the consistency condition (2.9), but this fact is not so easy to exploit directly. Instead we first explore the consequences of (3.10) and at the end confront the result with the consistency condition. By contracting the indices of the matrices (3.10) with A^I and B_I and taking a suitable linear combination, we can use the homogeneity of the function h (which is a fourth-order polynomial in A and B) to obtain the expression

$$\begin{aligned} h''(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star) &= -A^J B_I\, B^I{}_J(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star)+\tfrac12 A^I A^J\, C_{IJ}(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star)\\ &\quad +\tfrac12 B_I B_J\, D^{IJ}(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star)+\tfrac12(A\cdot\hat\beta_\star-B\cdot\hat\alpha_\star)(A\cdot\beta-B\cdot\alpha). \end{aligned} \tag{3.12}$$

Replacing α^I by A^I and β_I by B_I, and using once more the homogeneity of h, yields

$$\begin{aligned} (\hat\alpha_\star\cdot\partial+\hat\beta_\star\cdot\partial)h &= -\tfrac13 A^J B_I\, B^I{}_J(A,B,\hat\alpha_\star,\hat\beta_\star)\\ &\quad +\tfrac16 A^I A^J\, C_{IJ}(A,B,\hat\alpha_\star,\hat\beta_\star)+\tfrac16 B_I B_J\, D^{IJ}(A,B,\hat\alpha_\star,\hat\beta_\star). \end{aligned} \tag{3.13}$$

Resubstituting the above result into (3.12) shows that the matrices (3.10) can be decomposed as follows,

$$\begin{aligned}
C_{IJ}(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star) &= C_{IJK}(\hat\alpha_\star,\hat\beta_\star)\,\alpha^K + C^K{}_{IJ}(\hat\alpha_\star,\hat\beta_\star)\,\beta_K \\
D^{IJ}(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star) &= D^{IJ}{}_K(\hat\alpha_\star,\hat\beta_\star)\,\alpha^K + D^{IJK}(\hat\alpha_\star,\hat\beta_\star)\,\beta_K \\
B^I{}_J(\alpha,\beta,\hat\alpha_\star,\hat\beta_\star) &= -C^I{}_{JK}(\hat\alpha_\star,\hat\beta_\star)\,\alpha^K - D^{IK}{}_J(\hat\alpha_\star,\hat\beta_\star)\,\beta_K \\
&\quad -\tfrac{1}{2}(\hat\beta_\star\!\cdot\alpha + \beta\!\cdot\hat\alpha_\star)\,\delta^I_J - \hat\beta_{\star J}\,\alpha^I - \beta_J\,\hat\alpha^I_\star,
\end{aligned} \tag{3.14}$$

where the new three-index tensors introduced on the right-hand side are separately symmetric in upper and lower indices. This allows us to rewrite (3.13) as

$$\begin{aligned}
(\hat\alpha_\star\cdot\partial + \hat\beta_\star\cdot\partial)h &= \tfrac{1}{6}A^IA^JA^K\,C_{IJK}(\hat\alpha_\star,\hat\beta_\star) + \tfrac{1}{6}B_IB_JB_K\,D^{IJK}(\hat\alpha_\star,\hat\beta_\star) \\
&\quad +\tfrac{1}{2}B_IB_JA^K\,D^{IJ}{}_K(\hat\alpha_\star,\hat\beta_\star) + \tfrac{1}{2}A^JA^KB_I\,C^I{}_{JK}(\hat\alpha_\star,\hat\beta_\star) \\
&\quad +\tfrac{1}{2}(A\cdot\hat\beta_\star + B\cdot\hat\alpha_\star)\,(A\cdot B)\;.
\end{aligned} \tag{3.15}$$

The reader may wonder what we have gained here as (3.15) is still the most general expansion in terms of A^I and B_I. However, the matrices $B^I{}_J$, C_{IJ} and D^{IJ} associated with the duality transformations are restricted by the consistency equation (2.9); consequently similar conditions exist for the tensors on the right-hand side of (3.14), which, in addition, must be symmetric. To write down these restrictions is straightforward and we refrain from doing so. In section 4 we shall apply this strategy to the class of spaces related to the functions (2.12) (which do have at least one symmetry belonging to $\mathcal{V}_{-1/2}$) and show how it leads to a full determination of (3.15). Obviously, one may attempt to integrate (3.15) and in this way determine (part of) the function h. For the symmetric spaces we will determine h completely in this way. For other spaces we will determine the structure of h and completely determine the terms independent of z, using the duality invariance of h and (3.15). In contrast, the direct evaluation of h from its definition (2.31) is difficult (see, e.g. [19] where we presented the result for the simple case corresponding to Fig. 2, which required the use of a computer).

The assertion (*iv*) follows now directly from (3.14), as this equation cannot be satisfied for $C = D = B = 0$. Therefore the corresponding Kähler manifold must exhibit isometries associated with duality transformations.

Finally we explain under which conditions the **c** map preserves the homogeneity of the space. If a manifold is homogeneous due to a particular group of isometries, then a larger manifold in which the previous one is embedded is also homogeneous, provided the isometries of the submanifold can be extended to isometries of the bigger manifold and there exist additional isometries that act transitively on the additional coordinates. Now consider a special Kähler manifold, which is homogeneous due to the fact that certain duality transformations act transitively on the manifold. Under the **c** map these transformations are extended to isometries of the quaternionic manifold. Moreover there are the extra symmetries corresponding to $\underline{\epsilon}_0$, $\underline{\epsilon}_+$, $\underline{\alpha}_I$ and $\underline{\beta}^I$, which by (3.4) act transitively on the additional coordinates ϕ, σ, A^I and B_I. Therefore the image of the Kähler space under the **c** map is a homogeneous quaternionic space. In this respect it is important that, after dimensional reduction, the number of new symmetries is always larger than or equal to the number of new coordinates. We stress that the above arguments do not apply to the case where the Kähler manifold is homogeneous owing to isometries that are not symmetries of the full supergravity action. So far no examples of such a manifold are known.

Conversely, consider a manifold which is homogeneous. If there exists a subgroup of the isometries that acts transitively within a certain submanifold and leaves this submanifold invariant, then the submanifold is homogeneous. For the special quaternionic spaces we know from [19] that (3.4) comprises all the isometries of the quaternionic space. The corresponding special Kähler space, parametrized by the scalars z^A, is the submanifold defined by $\phi = 1$ and $\sigma = A^I = B_I = 0$. The duality transformations leave the submanifold invariant and are isometries of the Kähler metric. Moreover, as the quaternionic manifold was assumed to be homogeneous, any two points in the submanifold are related by an isometry transformation. Because only the action of the duality group remains inside the submanifold, any two points of the Kähler manifold must be related by a duality transformation. In other words, the duality transformations act transitively on the Kähler manifold. It then follows that the Kähler manifold is also homogeneous.

The same reasoning applies to the **r** map, where the duality transformations are replaced by transformations that involve vector and scalar fields and leave the $d = 5$ dimensional supergravity Lagrangian invariant.

4 Symmetries of d-spaces.

This section deals with the properties of special geometries based on the functions

$$F(X) = i d_{ABC} \frac{X^A X^B X^C}{X^0} \ , \tag{4.1}$$

where d_{ABC} are real coefficients. Some of the relevant material was already discussed in section 2. These functions describe all the special real manifolds. Under the **c** map and the **r** map they yield special Kähler and quaternionic spaces. The Kähler spaces were studied in [15].

After presenting some convenient notation and other preliminaries, we recall the symmetries for the real manifolds in subsection 4.1. Then we derive the conditions for the existence of hidden symmetries of the corresponding special Kähler manifold (these were already quoted in subsection 2.1) and summarize the results regarding the algebra in subsection 4.2. A large part of this section is devoted to the symmetry structure of the corresponding quaternionic spaces. This is done in subsection 4.3, where we give conditions for the existence of hidden symmetries, and describe the algebra of isometries and its main consequences. A summary of the conditions for existence of hidden symmetries of spaces based on (4.1) is given in subsection 4.4.

It is often convenient to decompose the complex fields z into real and imaginary parts,

$$z^A \equiv \tfrac{1}{2}(y^A - i x^A) \ . \tag{4.2}$$

The domain for the variables is restricted by the requirement that the kinetic terms for the scalars (2.13) and for the graviton are positive definite. This leads to the conditions

$$\begin{aligned} &dxxx \equiv d_{ABC} x^A x^B x^C > 0 \\ &3(dxx)_A (dxx)_B - 2(dx)_{AB}(dxxx) \ \ \text{a positive definite matrix.} \end{aligned} \tag{4.3}$$

To evaluate the quantities of interest we note that

$$N_{00} = \tfrac{1}{2}i(dzzz) + h.c. \; , \qquad N_{0A} = -\tfrac{3}{4}i(dzz)_A + h.c. \; , \qquad N_{AB} = \tfrac{3}{2}(dx)_{AB} \; , \tag{4.4}$$

so that

$$N_{0A} = -\tfrac{1}{2}N_{AB}(z+\bar{z})^B \; . \tag{4.5}$$

This implies

$$zN\bar{z} = \tfrac{1}{4}dxxx \; , \qquad (Nz)_A = \tfrac{3}{4}i(dxx)_A \; , \qquad \left(n^{-1}Nz\right)^A = -\tfrac{1}{2}ix^A \; , \tag{4.6}$$

where $\left(n^{-1}\right)^{AB} N_{BC} = \delta^A_C$.

An important property of these models is that, within the equivalence class of Kähler potentials $K(z,\bar{z}) \sim K(z,\bar{z}) + \Lambda(z) + \bar{\Lambda}(\bar{z})$, there are representatives which depend only on x. We will denote one such a representative by g

$$K(z,\bar{z}) \sim g(x) = \log dxxx \; . \tag{4.7}$$

Therefore the metric depends only on x, as we found already in (2.13), and is just the second derivative of g with respect to x. Using the notation where multiple x-derivatives of g are written as $g_{AB\ldots}$, the Kähler metric $g_{A\bar{B}}$ coincides with g_{AB}, without the need for distinguishing holomorphic and anti-holomorphic indices. All multiple derivatives are homogeneous functions of x and we get the following relations

$$\begin{aligned}
& g_A\, x^A = 3 \; ; \qquad g^{AB}\, g_B = -x^A \; , \\
& g_A = 3\, \frac{(dxx)_A}{dxxx} = -g_{AB}\, x^B \; , \\
& g_{AB} = 6\, \frac{(dx)_{AB}}{d\,xxx} - g_A\, g_B = -\tfrac{1}{2} g_{ABC}\, x^C \; , \\
& g_{ABC} = 6\, \frac{d_{ABC}}{dxxx} - 3 g_{(A}\, g_{BC)} - g_A\, g_B\, g_C \; ,
\end{aligned} \tag{4.8}$$

where g^{AB} is the inverse of g_{AB}, which equals

$$g^{AB} = \tfrac{1}{4}(dxxx)\left(n^{-1}\right)^{AB} - \tfrac{1}{2}x^A x^B \; . \tag{4.9}$$

The fourth derivative of g depends on lower derivatives. This explains why the curvature (2.15) depends only on triple derivatives of g.

The independent non-vanishing components of the Christoffel connection and the Riemann tensor of a Kähler manifold are generally given by

$$\Gamma^C_{AB} = g^{C\bar{C}}\, \partial_A g_{B\bar{C}} \; , \qquad R^A{}_{BC\bar{D}} = \partial_{\bar{D}} \Gamma^A_{BC} \; . \tag{4.10}$$

In the case at hand the connection is purely imaginary,

$$\Gamma^A_{BC} = -\Gamma^{\bar{A}}_{\bar{B}\bar{C}} = i g^{AD} g_{BCD} \; . \tag{4.11}$$

The (real) curvature equals

$$R^A{}_{BC\bar{D}} = R^{\bar{A}}{}_{\bar{B}\bar{C}D} = \frac{\partial}{\partial x^D}\left(g^{AE} g_{EBC}\right) \; . \tag{4.12}$$

From its homogeneity one proves

$$x^D R^A{}_{BC\bar{D}} = -g^{AE} g_{EBC} \; . \tag{4.13}$$

From (2.15) and (2.16) we can now verify the validity of (2.36) for the Kähler spaces of this section. Using the above expression for the connection (c.f.(4.10)), this leads to an expression for the tensor $\mathcal{C}$, homogeneous in x,

$$\mathcal{C}_{ABCD} = 3\, d_{ABC}\, g_D + \tfrac{9}{2}\, g_{FD(A}\, d_{BC)G}\, g^{FG} \; , \tag{4.14}$$

which is not manifestly symmetric. However, it becomes fully symmetric by using (4.8),

$$\mathcal{C}_{ABCD} = -6 d_{(ABC}\, g_{D)} + 27 (g^{GH} + x^G x^H)\, d_{H(AB}\, d_{CD)G}\, (dxxx)^{-1}. \tag{4.15}$$

Alternative and useful representations of $\mathcal{C}$, some not manifestly symmetric either, are easy to derive using (4.8),

$$\mathcal{C}_{ABCD} = \tfrac{1}{18} (dxxx)^2\, g_{AE}\, g_{BF}\, g_{CG} \frac{\partial}{\partial x^D} C^{EFG} = -6 g_{DF}\, E^F_{ABCE}\, x^E = 6 g_E\, E^E_{ABCD} \; . \tag{4.16}$$

where we used the definitions (2.16) and (2.21). There are many other relations between the various tensors. The following two are particularly useful,

$$x^A\, \mathcal{C}_{ABCD} = 0 \; , \tag{4.17}$$

$$\frac{\partial}{\partial x^E} \mathcal{C}_{ABCD} = 6 g_{EF}\, E^F_{ABCD} \; . \tag{4.18}$$

4.1 Symmetries of the real space.

The class of manifolds based on (4.1) comprise all special real spaces. The symmetries of the corresponding $d = 5$ supergravity theory were given in [13] (see subsection 2.2). The scalar fields h^A, satisfying (2.24), transform as

$$\delta h^A = \tilde{B}^A_B\, h^B \; , \tag{4.19}$$

where the matrices $\tilde{B}$ define an invariance of the tensor d_{ABC} (cf. (2.18)). However, the corresponding non-linear sigma model may possess extra invariances, which are not symmetries of the full action. In [14] we gave an example of a class of models where this is indeed the case. Under the **c** map these extra transformations are not preserved, so certain geometrical properties of the real spaces no longer hold for the corresponding special Kähler spaces. In general, only the isometries defined by (4.19) are relevant for the special Kähler and quaternionic manifolds. In [14], we proved that the isometries of the corresponding Kähler spaces are always symmetries of the full $d = 4$ supergravity theory. For the quaternionic spaces that emerge under the **c** map this result is obvious as all vector fields have then be converted to scalars. The algebra of the transformations (4.19) follows from the commutators of the matrices $\tilde{B}$ (cf. the analogous discussion of the algebra of the duality transformations at the beginning of section 3).

4.2 Symmetries of the Kähler space.

The non-linear sigma models corresponding to the Kähler manifolds that are in the image of the **c** map, couple to $d=4$ supergravity. The duality invariances of the corresponding actions were studied in [15]. As summarized in subsection 2.1 these invariances lead to isometries of the Kähler manifold expressed in terms of the parameters β, b^A, $\tilde{B}^A_B$ and a_A, appearing in (2.17). The isometries parametrized by β and b^A

$$\delta z^A = b^A - \tfrac{2}{3}\beta\, z^A\,, \tag{4.20}$$

exist irrespective of the form of the symmetric tensor d_{ABC} (simply because they find their origin in the dimensional reduction from five to four space-time dimensions (see subsection 2.2)). The matrices $\tilde{B}$ satisfy (2.18). The *hidden* invariances associated with a_A are realized for parameters satisfying (2.20). The derivation of the latter result starts from the master equation (2.9) on the matrices (2.17) that parametrize the duality transformations. This leads to

$$a_{(A}d_{BCD)} = -\tfrac{9}{4}D^{EF}d_{E(AB}d_{CD)F}\ . \tag{4.21}$$

Multiplying this equation with x^Cx^D and with $x^Bx^Cx^D$, one derives (2.19), which can be re-inserted into the above equation to yield (2.20). However, one must ensure that the D^{AB} are constant (as the matrices (2.16) are constant). Perhaps somewhat surprisingly, this is again a consequence of (2.20). Actually, it turns out that the following conditions are equivalent criteria for establishing the existence of the hidden Kähler symmetries,

$$E^E_{ABCD}a_E = 0 \Longleftrightarrow \mathcal{C}_{ABCD}\, g^{DE}\, a_E = 0 \Longleftrightarrow \frac{\partial}{\partial x^D}C^{ABC}a_C = 0\ . \tag{4.22}$$

The fact that the first condition implies the second and the third, follows directly from (4.16). The second and third condition are equivalent and imply that $a_E E^E_{ABCD}x^D = 0$, again because of (4.16). Because $C^{ABC}a_C$ is constant, also $a_E E^E_{ABCD}$ is constant. Since we know that this constant should vanish upon multiplication with x^D, it should itself be zero, which is precisely the first condition (4.22). This proves (4.22). Finally we remind the reader that the conditions (4.22) are also equivalent to $R^A{}_{BC}{}^D a_D$ being constant. The space is symmetric if and only if the tensors E^E_{ABCD} and $\mathcal{C}_{ABCD}$ vanish; in that case $R^A{}_{BC}{}^D$ is thus constant (as well as covariantly constant in view of (2.36)).

The algebra of the symmetries of the Kähler space was obtained in [15]. Apart from the commutator given in the previous subsection, the non-zero commutators are

$$\begin{array}{ll} [\underline{\beta},\, \underline{b}_A] = \tfrac{2}{3}\underline{b}_A\ ; & [\underline{\beta},\, \underline{a}^A] = -\tfrac{2}{3}\underline{a}^A\ , \\ [\omega^i\underline{\omega}_i,\, \underline{b}_B] = -\tilde{B}^A{}_B(\omega)\, \underline{b}_A\ ; & [\omega^i\underline{\omega}_i,\, \underline{a}^A] = \tilde{B}^A{}_B(\omega)\, \underline{a}^B\ , \\ [\underline{b}_B,\, \underline{a}^A] = \delta^A_B\, \underline{\beta} + \omega_B^{A\,i}\underline{\omega}_i\ , & \end{array} \tag{4.23}$$

where $\omega_B^{A\,i}$ are the transformation parameters corresponding to

$$\tilde{B}^C{}_D(\omega^A_B) = R^A{}_{BD}{}^C + \tfrac{2}{3}\delta^C_D\,\delta^A_B\ . \tag{4.24}$$

These commutators clearly exhibit the decomposition (2.26). From the algebra we deduce that the existence of a symmetry associated with parameters a_A implies the existence of duality invariances of the form

$$\tilde{B}^A{}_B(a,b) = \tfrac{4}{3}a_C C^{ACE}d_{BDE}\, b^D - \tfrac{1}{3}\delta^A_B\, b^C a_C - b^A a_B\ , \tag{4.25}$$

with b^A arbitrary. This follows from taking the commutator of the hidden symmetry with any one of the variations parametrized by b^A (which are always realized).

4.3 Symmetries of the quaternionic space.

All special quaternionic spaces possess the symmetries of (2.29). The existence of possible hidden symmetries depends on existence of solutions of (2.43) and (2.44). Here we investigate the implications of these equations for the spaces based on the functions (4.1). These quaternionic spaces are thus in the image of the **c**∘**r** map. Using the results of the previous subsections we first discuss a simpler version of the equations. Subsequently, we exploit the results of section 3 based on the algebra of isometries, which are very powerful. Before turning to the proofs, let us list a number of important results. We shall show that $\hat\beta_0$ is always associated with a symmetry, that the conditions for $\hat\alpha^A$ and $\hat\beta_A$ can be written as

$$E^E_{ABCD}\hat\alpha^D = 0 \,, \tag{4.26}$$

$$E^E_{ABCD}\hat\beta_E = 0 \,, \tag{4.27}$$

and that $\hat\alpha_0$ is only associated with a symmetry iff the space is symmetric ($E^E_{ABCD}=0$). Below we prove these results and exhibit how the existence of certain symmetries implies in turn the existence of other symmetries. Furthermore we exploit the results of section 3 to obtain information on the function h defined in (2.31). Some of these results were already reported in [19].

First we note that $\hat\xi^A$ defined in (2.45) simplifies to

$$\hat\xi^A = 4(dxxx)^{-1}\left(g^{AB}\xi_B + ix^A\xi_I z^I\right) \,, \tag{4.28}$$

where we used relations derived at the beginning of this section and $\Delta = -\frac{1}{8}(dxxx)$. Using (4.17) and the homogeneity of the tensor $\mathcal{C}$, (2.43) and (2.44) take the form

$$\mathcal{C}_{ABCD}g^{DE}\bar\xi_E = 0 \,, \qquad \xi_F g^{FE}\frac{\partial}{\partial x^E}\mathcal{C}_{ABCD} = 0 \,. \tag{4.29}$$

Furthermore (2.41) becomes

$$\xi_A = \hat\beta_A - 3d_{ABC}z^C\hat\alpha^B + \tfrac{3}{2}(dzz)_A\hat\alpha^0 \,, \tag{4.30}$$

which, unlike $\mathcal{C}$, depends on both x^A and y^A; expansion in y thus leads to a number of independent equations when substituted into (4.29). The equations for $\hat\alpha^A$ and $\hat\beta_A$ that follow from (4.29), are

$$\begin{aligned}
&\mathcal{C}_{ABCD}\, g^{DE}\,\hat\beta_E = 0 \,; &&\hat\beta_F\, g^{FE}\frac{\partial}{\partial x^E}\mathcal{C}_{ABCD} = 0 \,,\\
&\mathcal{C}_{ABCD}\, g^{DE}\, d_{EFG}\,\hat\alpha^F = 0 \,; &&\hat\alpha^H\, d_{HGF}\, g^{FE}\frac{\partial}{\partial x^E}\mathcal{C}_{ABCD} = 0.
\end{aligned} \tag{4.31}$$

Furthermore, we can only have a symmetry associated with the parameter $\hat\alpha^0$ provided that $\mathcal{C}=0$. This implies that the corresponding Kähler space is symmetric (and, as it turns out from explicit application of the **c** map, so is the corresponding quaternionic space).

None of the above equations depend on $\hat\beta_0$, so that the hidden symmetry corresponding to this parameter is *always* realized. Subsequently consider the equations (4.31) proportional to $\hat\beta$. It turns out that the first equation coincides with the second condition for the hidden Kähler symmetry characterized by the parameter a_A in (4.22), while according to (4.18) and (4.22), the second equation is just equivalent to the first. This establishes that the condition for the existence of hidden quaternionic symmetries associated with the parameters $\hat\beta_A$ coincides with the condition for the hidden Kähler symmetries parametrized by a_A, which, again according to (4.22), is precisely (4.27). Consequently, every hidden Kähler symmetry implies the existence of a hidden quaternionic symmetry associated with parameters $\hat\beta_A$.

Following the same arguments, the equations (4.31) for the symmetries associated with the parameters $\hat\alpha^A$ are that the parameters $\hat\beta_A = d_{ABC}\hat\alpha^B$ satisfy (4.27) and thus constitute hidden symmetries for all index values for C. According to (4.22) this means that $E^E_{ABCD}\, d_{EFG}\,\hat\alpha^F = 0$. The latter proves that parameters $\hat\alpha$ satisfying (4.26) are symmetries. Indeed, the identity

$$E^E_{ABCD}\, d_{FGE} + 2E^E_{F(ABC}\, d_{D)GE} - 3E^E_{FG(AB}\, d_{CD)E} = 0\ , \tag{4.32}$$

multiplied with $\hat\alpha^F$, shows that for parameters satisfying (4.26), $\hat\beta_A = d_{ABC}\hat\alpha^B$ are symmetries for all index values of C. This proves that (4.26) is a sufficient condition. Its necessity will be derived shortly.

Many of the above results and their implications can be obtained from the algebra of isometries discussed in section 3. The fact that the $\hat\alpha^0$ symmetry does not exist unless the space is symmetric, can be understood from the result (i) and the existence of the $\hat\beta_0$ symmetry. For symmetric spaces all hidden symmetries are realized; therefore we will not discuss them further and ignore the $\hat\alpha^0$ symmetry. For non-symmetric spaces the independent parameters $\hat\alpha^A$ and $\hat\beta_A$ are subject to the constraint $\hat\alpha^A\,\hat\beta_A = 0$. The result ($ii$) now takes the following form. From (3.7) and (3.8) we conclude that if we know of the existence of symmetries with parameters $\hat\alpha^I_\star$, $\hat\beta_{\star I}$, then hidden symmetries must be realized with the following parameters $\hat\alpha^I$, $\hat\beta_I$,

$$\begin{pmatrix}\hat\alpha^0\\ \hat\alpha^A\\ \hat\beta_0\\ \hat\beta_A\end{pmatrix} = \begin{pmatrix}\beta & a_B & 0 & 0\\ b^A & \tilde B^A{}_B + \frac13\beta\delta^A_B & 0 & \frac49 C^{ABC}a_C\\ 0 & 0 & -\beta & -b^B\\ 0 & 3d_{ABC}b^C & -a_A & -\tilde B^B{}_A - \frac13\beta\delta^A_B\end{pmatrix}\begin{pmatrix}\hat\alpha^0_\star\\ \hat\alpha^A_\star\\ \hat\beta_{\star 0}\\ \hat\beta_{\star A}\end{pmatrix}\ , \tag{4.33}$$

where the parameters β, b^A, $\tilde B^A_B$ and a_A parametrize duality transformations. We list three consequences of this formula:

(i). We established that the symmetry associated with the parameter $\hat\beta_{\star 0}$ is always realized. Therefore, if there are duality transformations characterized by parameters a_A, there must be extra symmetries with $\hat\beta_A \propto a_A$. This explains why the conditions for the existence of $\hat\beta_A$ symmetries are the same as those for a_A symmetries, as derived above.

(ii). If $\hat\alpha^A_\star$ corresponds to an isometry, so does $\tilde B^A_B\,\hat\alpha^B_\star$. By using the duality transformations proportional to b^A (which are always realized) we conclude that the parameters $\hat\beta_A \propto d_{ABC}\hat\alpha^B_\star$ should correspond to an isometry as well, in accord with the result found above.

(iii). If there are symmetries corresponding to the parameters $\hat\beta_{\star B}$ then there should also be symmetries corresponding to the parameters $\tilde B^B{}_A\,\hat\beta_{\star B}$. Furthermore we can make use of the fact that for every duality transformations proportional to a_A, there exist a hidden quaternionic symmetry associated with the parameters $\hat\beta_{\star A} \propto a_A$. Then there must be an isometry associated with the parameters

$$\hat\alpha^A = C^{ABC}\,\hat\beta_{\star B}\hat\beta_{\star C} \tag{4.34}$$

for all independent parameters $\hat\beta_{\star A}$ associated with a hidden quaternionic symmetry.

We now explore the commutator (3.9) discussed in section 3. The resulting duality transformation, parametrized by (3.14), should be in accord with the general form of the duality transformations of the Kähler spaces based on (4.1), specified in (2.17) and (2.19), for arbitrary α and β. This requirement leads to surprisingly restrictive conditions for the coefficients in (3.14). The non-vanishing coefficients are given by

$$\begin{array}{ll} C_{ABC}(\hat\alpha,\hat\beta) = -3d_{ABC}\hat\beta_0\ , & D^{ABC}(\hat\alpha,\hat\beta) = \frac{4}{9}C^{ABC}\hat\alpha^0\ , \\ C^C{}_{AB}(\hat\alpha,\hat\beta) = -\frac{4}{3}d_{ABD}C^{CDE}\hat\beta_E\ , & D^{AB}{}_C(\hat\alpha,\hat\beta) = -\frac{4}{3}C^{ABD}d_{CDE}\hat\alpha^E\ , \\ C^0{}_{AB}(\hat\alpha,\hat\beta) = -3d_{ABC}\hat\alpha^C\ , & D^{AB}{}_0(\hat\alpha,\hat\beta) = \frac{4}{9}C^{ABC}\hat\beta_C\ , \end{array} \tag{4.35}$$

while the parameters of the duality transformations in the commutator (3.9) are equal to

$$\begin{aligned} \beta(\alpha,\beta,\hat\alpha,\hat\beta) &= -\tfrac{1}{2}\hat\beta_I\alpha^I - \hat\beta_0\alpha^0 - \tfrac{1}{2}\beta_I\hat\alpha^I - \beta_0\hat\alpha^0 \\ b^A(\alpha,\beta,\hat\alpha,\hat\beta) &= -\tfrac{4}{9}C^{ABC}\beta_B\hat\beta_C - \hat\beta_0\alpha^A - \beta_0\hat\alpha^A \\ a_A(\alpha,\beta,\hat\alpha,\hat\beta) &= 3d_{ABC}\alpha^B\hat\alpha^C - \hat\alpha^0\beta_A - \alpha^0\hat\beta_A \\ \tilde B^A{}_B(\alpha,\beta,\hat\alpha,\hat\beta) &= \tfrac{4}{3}C^{ACD}d_{BCE}(\hat\beta_D\alpha^E + \beta_D\hat\alpha^E) \\ &\quad -\tfrac{1}{3}\delta^A_B(\hat\beta_C\alpha^C + \beta_C\hat\alpha^C) - \hat\beta_B\alpha^A - \beta_B\hat\alpha^A. \end{aligned} \tag{4.36}$$

The above parameters should satisfy (2.18) and (2.20) for all α^I and β_I. (Observe that $\hat\beta_0$ does not appear in the expressions for a_A and $\tilde B^A{}_B$, so it is left unrestricted as it should.) This gives rise to the equations

$$\hat\alpha^0 E^E_{ABCD} = E^E_{ABCD}\,d_{EFG}\,\hat\alpha^F = E^E_{ABCD}\,\hat\beta_E = E^E_{ABCD}\,\hat\alpha^D = 0\ . \tag{4.37}$$

This confirms the previous results and finally establishes the necessity for the condition (4.26).

Finally we exploit the result 3 on page 17 to further determine the function h. Using (4.35) we may write (3.15) as

$$\begin{aligned} (\hat\alpha^I\partial_I + \hat\beta_I\partial^I)h &= \hat\beta_0\left\{-\tfrac{1}{2}d_{ABC}\,A^AA^BA^C + \tfrac{1}{2}A^0A^IB_I\right\} \\ &\quad +\hat\alpha^0\left\{\tfrac{2}{27}C^{ABC}\,B_AB_BB_C + \tfrac{1}{2}A^IB_IB_0\right\} \\ &\quad +\hat\beta_A\left\{\tfrac{2}{9}A^0\,C^{ABC}\,B_BB_C - \tfrac{2}{3}A^DA^E\,d_{BDE}\,C^{ABC}\,B_C + \tfrac{1}{2}A^AA^IB_I\right\} \\ &\quad +\hat\alpha^A\left\{-\tfrac{2}{3}A^C\,d_{ABC}\,C^{BDE}\,B_DB_E - \tfrac{2}{3}A^BA^C\,d_{ABC}\,B_0 + \tfrac{1}{2}A^IB_IB_A\right\}\ , \end{aligned} \tag{4.38}$$

for the independent parameters $\hat\alpha$ and $\hat\beta$ that are associated with a hidden symmetry. We recall that the right-hand side of this equation does not depend on z^A and $\bar z^A$. In the case at hand, we know that the symmetry associated with $\hat{\beta}_0$ is always realized, so that the derivative of h with respect to B_0 equals

$$\partial^0 h = -\tfrac{1}{2} d_{ABC} A^A A^B A^C + \tfrac{1}{2} A^0 A^I B_I \; . \tag{4.39}$$

This determines the function h up to a quartic polynomial in A^0, A^A and B_A, with z- and $\bar{z}$-dependent coefficient functions. However, h should be invariant under duality transformations. In this case we know that the transformations parametrized by β and b^A are always realized. For the fields A^0, A^A, B_0, B_A, these transformations can be read off immediately from (4.33), while for z and $\bar{z}$ they are given in (4.20). To further determine the function, such that the result is manifestly invariant under these duality transformations, we express it in terms of fields that are invariant under the duality transformations proportional to b^A, namely x^A, A^0 and

$$\begin{aligned} \tilde{A}^A &= A^A - \tfrac{1}{2} A^0\, y^A \;, \\ \tilde{B}_A &= B_A - \tfrac{3}{2} (d\,y)_{AB}\, A^B + \tfrac{3}{8} (d\,yy)_A\, A^0 \;, \\ \tilde{B}_0 &= B_0 + \tfrac{1}{2} y^A\, B_A - \tfrac{3}{8} (d\,yy)_A\, A^A + \tfrac{1}{16} (d\,yyy)\, A^0 \;, \end{aligned} \tag{4.40}$$

where y^A and x^A are proportional to the real and imaginary part of z^A (cf. (4.2)). Expressing a function in terms of these variables and y^A, the invariance of h under the b^A transformations implies that there is no explicit dependence on y^A, so we may write $h(A^0, \tilde{A}^A, \tilde{B}_0, \tilde{B}_A, x^A)$. Because $\frac{\partial}{\partial B^0} h = \frac{\partial}{\partial \tilde{B}^0} h$, and the right-hand side of (4.39) does not change under the replacement of A^A, B_A and B_0 by $\tilde{A}^A$, $\tilde{B}_A$ and $\tilde{B}_0$ (due to the fact that this expression is invariant under the duality transformations proportional to b^A), we have thus determined the dependence of h on $\tilde{B}^0$. The other terms in h are quartic in A^0, $\tilde{A}^A$ and $\tilde{B}_A$, and there are 15 combinations of these. Under duality transformations parametrized by the parameter β these fields scale with weight 1, $\frac{1}{3}$ and $-\frac{1}{3}$, respectively, while x^A carries weight $-\frac{2}{3}$. Each coefficient function of x is thus homogeneous of a suitable degree. There appear functions of all degrees from -2 to 6. For instance, the contribution to h that is of zeroth degree in x, depends on two such functions, $h^{ABC}(x)$ and $h^{AB}_{CD}(x)$,

$$\begin{aligned} h^{(0)} &= -\tfrac{1}{2} d_{ABC}\, \tilde{A}^A \tilde{A}^B \tilde{A}^C\, \tilde{B}_0 + \tfrac{1}{2} A^0\, \tilde{A}^A\, \tilde{B}_A\, \tilde{B}_0 + \tfrac{1}{4} (A^0)^2 (\tilde{B}_0)^2 \\ &\quad + A^0\, \tilde{B}_A\, \tilde{B}_B\, \tilde{B}_C\, h^{ABC}(x) + \tilde{A}^A\, \tilde{A}^B\, \tilde{B}_C\, \tilde{B}_D\, h^{CD}_{AB}(x) \; . \end{aligned} \tag{4.41}$$

This function is manifestly invariant under duality transformations associated with the parameters b^A and β and satisfies (4.39). The remaining functions are $\tilde{B}_0$ independent and of non-zero degree in x.

When there are additional symmetries known proportional to parameters $\hat{\beta}_A$ and/or $\hat{\alpha}^I$ one can again follow the same strategy and use (4.38) to further restrict the function h. A special case is that of the symmetric space, where the maximal number of symmetries is realized, so that (4.38) holds for all $\hat{\alpha}^I$ and $\hat{\beta}_I$ separately. In that case all first-order derivatives with respect to A_I and B_I are known and we can integrate the result to find the function h,

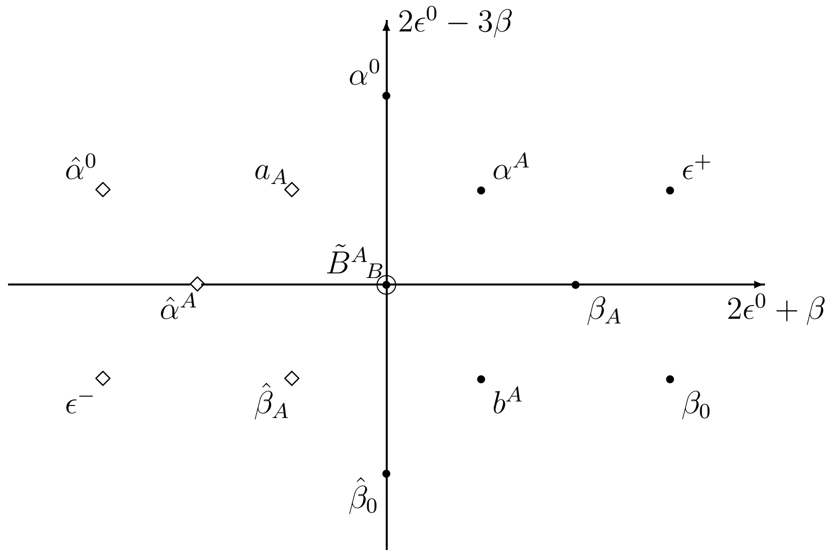
$$\begin{aligned} h &= -\tfrac{1}{2} d_{ABC}\, A^A\, A^B\, A^C\, B_0 + \tfrac{1}{2} A^0\, A^A\, B_A\, B_0 + \tfrac{1}{4} (A^0)^2 (B_0)^2 \\ &\quad + \tfrac{2}{27} A^0\, B_A\, B_B\, B_C\, C^{ABC} - \tfrac{1}{3} A^A\, A^B\, B_C\, B_D\, d_{ABE} C^{ECD} + \tfrac{1}{4} A^A\, B_A\, A^B\, B_B . \end{aligned} \tag{4.42}$$

One can check that this result is invariant under all duality transformations. In particular it is invariant under the duality transformations proportional to the parameters b^A. This implies that the replacement of A^A, B_A and B_0 by $\tilde{A}^A$, $\tilde{B}_A$ and $\tilde{B}_0$ leaves the function unaffected. To show this it is important that the tensor E^E_{ABCD} vanishes. For $n = 1$ with $d_{111} = C^{111} = 1$ the above formula for h confirms the result of a computer calculation presented in [19].

4.4 Summary

For the d-spaces the symmetries that are realized for the various types of special manifolds are summarized in Fig. 3 and in Table 1. The figure can be compared to Fig. 2, which

Figure 3: Roots of quaternionic d-spaces in the β-ϵ^0 plane



corresponds to the case where the isometry algebra of the quaternionic space equals G_2, except that we have changed the axes. The corresponding quaternionic space has rank 2, and is a d-space, i.e., it is in the image of the **c**∘**r** map.

The symmetries with $2\epsilon^0 + \beta \geq 0$ that are listed above the first horizonal line in the table, are realized for arbitrary coefficients d_{ABC}. All those roots are located on the right half-plane in the figure.

The symmetries associated with $\tilde{B}^A_B$, which are the remaining symmetries with $2\epsilon^0 + \beta = 0$, are listed in the table between the next two horizontal lines. The existence of these symmetries is related to solutions of the equation

$$d_{D(AB}\tilde{B}^D{}_{C)} = 0 \ . \tag{4.43}$$

The latter represent all invariances of the full $d = 5$ supergravity theory that contains a non-linear sigma model with the real space as a target space. It was shown in [14] that the real manifold can have additional isometries, which are not preserved by the supergravity interactions and therefore do not reappear as isometries of the corresponding Kähler and quaternionic manifolds. Therefore those symmetries are not listed here.

Table 1: Symmetries of d-spaces

real space	Kähler space	quaternionic space	$\underline{\beta}$	$\underline{\epsilon}_0$	$2\underline{\epsilon}_0+\underline{\beta}$	$2\underline{\epsilon}_0-3\underline{\beta}$
		ϵ^+	0	1	2	2
		α^0	-1	$\frac{1}{2}$	0	4
		β_0	1	$\frac{1}{2}$	2	-2
		α^A	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	2
		β_A	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{4}{3}$	0
	b^A	b^A	$\frac{2}{3}$	0	$\frac{2}{3}$	-2
	$\underline{\beta}$	$\underline{\beta},\ \underline{\epsilon}^0$	0	0	0	0
		$\hat{\beta}_0$	1	$-\frac{1}{2}$	0	-4
$\tilde{B}^A{}_B$	$\tilde{B}^A{}_B$	$\tilde{B}^A{}_B$	0	0	0	0
	a_A	a_A	$-\frac{2}{3}$	0	$-\frac{2}{3}$	2
		$\hat{\beta}_A$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	-2
		$\hat{\alpha}^A$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{4}{3}$	0
		$\hat{\alpha}^0$	-1	$-\frac{1}{2}$	-2	2
		ϵ^-	0	-1	-2	-2

The Cartan subalgebra of the $\tilde{B}^A{}_B$ transformations gives rise to additional dimensions of the root diagram, which are not depicted in Fig. 3. Their presence depends on the particular choice for the d-coefficients. In the next section, these generators will be specified for the homogeneous d-spaces. The Cartan subalgebra of the corresponding Kähler space consists of the Cartan subalgebra of the real space complemented by the generator $\underline{\beta}$. The Cartan subalgebra of the quaternionic space consists of the Kähler Cartan subalgebra extended with the generator $\underline{\epsilon}^0$. Hence the rank of the isometry algebra always increases by one unit.

The symmetries with $2\epsilon^0+\beta=-\frac{2}{3}$ are related to the solutions of

$$E^E_{ABCD}\,a_E = E^E_{ABCD}\hat{\beta}_E = 0\ . \tag{4.44}$$

Therefore, the condition determines the existence of hidden symmetries for both the Kähler and the quaternionic manifold. In Fig. 3 one can see that the $\hat{\beta}_0$ symmetry, which is always realized, relates the a_A to the $\hat{\beta}_A$ symmetries.

The symmetries with $2\epsilon^0+\beta=-\frac{4}{3}$ are related to solutions of

$$E^E_{ABCD}\hat{\alpha}^D = 0\ . \tag{4.45}$$

Finally, the symmetries with $2\epsilon^0+\beta=-2$ exist only for the symmetric spaces, which means that

$$E^E_{ABCD} = 0\ . \tag{4.46}$$

The existence of symmetries with a certain (negative) eigenvalue of $2\underline{\epsilon}_0 + \underline{\beta}$ implies always that symmetries exist with an eigenvalue that is $\frac{2}{3}$ larger. For instance, the existence of a_A symmetries implies the existence of a $\tilde{B}$ symmetry of the form (4.25) for arbitrary b^A, and presence of $\hat{\alpha}^A$ symmetries implies that $\hat{\beta}_A = d_{ABC}\hat{\alpha}^B$ is a symmetry for all C.

5 Homogeneous d-spaces.

For a homogeneous manifold the isometry group acts transitively, which means that every two points of the manifold are related by an isometry transformation. Therefore the manifold itself is (locally) determined by the isometry group. In this section we want to study the isometry group G for homogeneous spaces based on the coefficients d_{ABC}. Again these spaces come in three varieties, namely real, Kähler and quaternionic. In [14] they were called 'very special'. The homogeneous quaternionic spaces consist of the (symmetric) quaternionic projective spaces, the symmetric spaces that are related to the projective complex spaces via the **c** map and a subset of the d-spaces. The corresponding Kähler and real spaces are homogeneous as well, according to the arguments presented at the end of section 3. For the Kähler spaces we know from [14] that there are no other homogeneous spaces based on d_{ABC} coefficients, while all symmetric special Kähler spaces are contained in this class according to [27]. Whether or not there are real special spaces that are homogeneous and do not lead to homogeneous Kähler and quaternionic spaces via the **r** and the **c** map, is not known. For such spaces the transitive symmetry (sub)group cannot be extended to a symmetry of the corresponding supergravity action; at present no example of such a space is known.[5] In this section we discuss all homogeneous quaternionic spaces based on d_{ABC} coefficients and their real and Kähler partners.

The d_{ABC} coefficients that give rise to homogeneous spaces were classified in [17]. The solutions are denoted by $L(q,P)$, where q and P are integers with $q \geq -1$ and $P \geq 0$. For q equal to a multiple of 4, there exist additional solutions denoted by $L(4m,P,\dot{P})$, with $m \geq 0$ and $P,\dot{P} \geq 1$. Here $L(4m,P,\dot{P}) = L(4m,\dot{P},P)$. Some of the corresponding spaces were assigned various names in the literature [21, 16, 17], which are given in in Table 2, where the spaces discovered in [17] are indicated by a $\star$. The symmetric spaces are the three varieties corresponding to $L(-1,0)$, $L(0,P)$, $L(1,1)$, $L(2,1)$, $L(4,1)$, and $L(8,1)$, and the real spaces corresponding to $L(-1,P)$. For those cases the isometry group G is given. For the non-symmetric spaces the isometry group G is not semisimple; the isotropy group H is always its maximal compact subgroup. The aim of this section is to clarify the structure of these isometry groups G.

The d_{ABC} coefficients for these spaces can be specified as follows (we use here the parametrization chosen in the last section of [17], which is different from the so-called 'canonical parametrization'). First we decompose the coordinates h^A into h^1, h^2, h^μ and h^i, where the indices μ and i run over $q+1$ and r values, respectively. Hence we have $n = 3 + q + r$. The non-zero components of d_{ABC} are

$$d_{122} = 1\ ; \qquad d_{1\mu\nu} = -\delta_{\mu\nu}\ ; \qquad d_{2ij} = -\delta_{ij}\ ; \qquad d_{\mu ij} = \gamma_{\mu ij}\ , \tag{5.1}$$

[5] In [14] examples were discussed of real special spaces whose isometries were not all contained in the invariance group of the supergravity action; in spite of that the latter group still acted transitively on the space.

$C(h)$	real	Kähler	quaternionic
$L(-1,0)$	$SO(1,1)$	$\left[\frac{SU(1,1)}{U(1)}\right]^2$	$\frac{SO(3,4)}{(SU(2))^3}$
$L(-1,P)$	$\frac{SO(P+1,1)}{SO(P+1)}$	$\star$	$\star$
$L(0,0)$	$[SO(1,1)]^2$	$\left[\frac{SU(1,1)}{U(1)}\right]^3$	$\frac{SO(4,4)}{SO(4)\otimes SO(4)}$
$L(0,P)$	$\frac{SO(P+1,1)}{SO(P+1)} \otimes SO(1,1)$	$\frac{SU(1,1)}{U(1)} \otimes \frac{SO(P+2,2)}{SO(P+2)\otimes SO(2)}$	$\frac{SO(P+4,4)}{SO(P+4)\otimes SO(4)}$
$L(0,P,\dot{P})$	$Y(P,\dot{P})$	$K(P,\dot{P})$	$W(P,\dot{P})$
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$
$L(4m,P,\dot{P})$	$\star$	$\star$	$\star$
$L(1,1)$	$\frac{S\ell(3,\mathbb{R})}{SO(3)}$	$\frac{Sp(6)}{U(3)}$	$\frac{F_4}{USp(6)\otimes SU(2)}$
$L(2,1)$	$\frac{S\ell(3,\mathbb{C})}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	$\frac{E_6}{SU(6)\otimes SU(2)}$
$L(4,1)$	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6)\otimes U(1)}$	$\frac{E_7}{SO(12)\otimes SU(2)}$
$L(8,1)$	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes U(1)}$	$\frac{E_8}{E_7\otimes SU(2)}$

Table 2: Homogeneous special real spaces and their corresponding Kähler and quaternionic spaces. The rank of the real spaces is equal to 1 (above the line) or 2 (below the line). The rank of the corresponding Kähler and quaternionic manifolds is increased by one or two units, respectively. The integers P, $\dot{P}$, q and m can take all values ≥ 1.

so that $C(h)$ can be written as

$$C(h) = 3\Big\{h^1\,(h^2)^2 - h^1\,(h^\mu)^2 - h^2\,(h^i)^2 + \gamma_{\mu ij}\,h^\mu\,h^i\,h^j\Big\}\ . \tag{5.2}$$

The coefficients $\gamma_{\mu ij}$ are the generators of a $(q+1)$-dimensional real Clifford algebra with positive signature, denoted by $\mathcal{C}(q+1,0)$,

$$\gamma_{\mu ik}\,\gamma_{\nu kj} + \gamma_{\nu ik}\,\gamma_{\mu kj} = 2\delta_{ij}\delta_{\mu\nu}\ . \tag{5.3}$$

These Clifford algebras and some of their properties are given in Table 3 [28]. The irreducible representations for a given q are unique, except when the Clifford module consists of a direct sum of two factors. As shown in Table 3 this is the case for $q=0$ mod 4, where there exist two inequivalent irreducible representations. This implies that, for $q\neq 0$ mod 4, the gamma matrices are unique once we specify the number of irreducible representations. This number is denoted by P, so that $L(q,P)$ determines the gamma matrices in this case, and thus the d_{ABC} coefficients. For $q=0$ mod 4 the representations γ_μ and $-\gamma_\mu$ are not equivalent, and a reducible representation is characterized by the multiplicity of each of these representations, P and $\dot{P}$. Of course, an overall sign change of all the gamma matrices can be absorbed into a redefinition of h^μ in (5.2). This is the reason why $L(4m,P,\dot{P}) = L(4m,\dot{P},P)$. If the representation consists of copies of one version of the irreducible representations, then we denote it by $L(4m,P)$. The dimension

q	$q+1$	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathcal{S}_q(P,\dot{P})$
-1	0	$\mathbb{R}$	1	$SO(P)$
0	1	$\mathbb{R}\oplus\mathbb{R}$	1	$SO(P)\otimes SO(\dot{P})$
1	2	$\mathbb{R}(2)$	2	$SO(P)$
2	3	$\mathbb{C}(2)$	4	$U(P)$
3	4	$\mathbb{H}(2)$	8	$U(P,\mathbb{H})\equiv USp(2P)$
4	5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	8	$USp(2P)\otimes USp(2\dot{P})$
5	6	$\mathbb{H}(4)$	16	$U(P,\mathbb{H})\equiv USp(2P)$
6	7	$\mathbb{C}(8)$	16	$U(P)$
7	8	$\mathbb{R}(16)$	16	$SO(P)$
$n+7$	$n+8$	$\mathbb{R}(16)\otimes\mathcal{C}(n,0)$	$16\ \mathcal{D}_n$	as for $q+1=n$

Table 3: Real Clifford algebras $\mathcal{C}(q{+}1,0)$. Here $\mathbf{F}(n)$ stands for $n\times n$ matrices with entries over the field $\mathbf{F}$, while $\mathcal{D}_{q+1}$ denotes the real dimension of an irreducible representation of the Clifford algebra. $\mathcal{S}_q(P,\dot{P})$ is the metric preserving group in the centralizer of the Clifford algebra in the $(P+\dot{P})\mathcal{D}_{q+1}$-dimensional representation.

of an irreducible representation is denoted by $\mathcal{D}_{q+1}$ and given in table 3. We thus have $r = P\,\mathcal{D}_{q+1}$, or $r=(P+\dot{P})\,\mathcal{D}_{q+1}$, and thus[6]

$$n = 3+q+(P+\dot{P})\,\mathcal{D}_{q+1}\ . \tag{5.4}$$

5.1 Symmetries of the Clifford algebra.

The isometry group of the various manifolds contains the linear transformations of the coordinates h^A that leave the cubic polynomial (5.2) invariant. A special subgroup of these invariance transformations corresponds to the those that leave the Clifford algebra invariant and preserve the metrics $\delta_{\mu\nu}$ and δ_{ij}. In [17] it was suggested that these transformations are relevant for the homogeneous spaces that occur in matter-coupled supergravity actions in 6 dimensions; these couplings would then be determined by the Clifford algebra.

The symmetry transformations of the γ matrices were discussed in [17]. The symmetries of $\delta_{\mu\nu}$ form the rotation group $SO(q{+}1)$. They leave the γ matrices invariant when acting simultaneously on the spinor and vector coordinates, labelled by i and μ, respectively. On the spinor indices the rotations act according to the cover group. Besides there can be additional invariances that act exclusively in spinor space and commute with the gamma matrices and thus with the corresponding representation of the Clifford algebra. These are the antisymmetric matrices S_{ij} determined by

$$[\gamma_\mu, S]=0\ . \tag{5.5}$$

They are the metric-preserving elements of the centralizer of the Clifford algebra representation, which are denoted by[7] $\mathcal{S}_q(P,\dot{P})$, and listed in Table 3. The invariance group of

[6] For $L(q,P)$ one must take $\dot{P}=0$.

[7] Both the group and the corresponding Lie algebra will be denoted by $\mathcal{S}_q(P,\dot{P})$.

$\gamma_{\mu ij}$, δ_{ij} and $\delta_{\mu\nu}$ is thus

$$SO(q+1) \otimes \mathcal{S}_q(P, \dot{P}) \;, \tag{5.6}$$

which acts according to

$$\begin{aligned} \delta h^\mu &= A_{\mu\nu}\, h^\nu \;, \\ \delta h^i &= \tfrac{1}{4} A_{\mu\nu}\, (\gamma_\mu \gamma_\nu)^i{}_j\, h^j + S^i{}_j\, h^j \;, \end{aligned} \tag{5.7}$$

where both $A_{\mu\nu}$ and S_{ij} are anti-symmetric matrices. The dimension of the above invariance group equals

$$\tfrac{1}{2} q(q+1) + \tfrac{1}{2} e_q P(P+1) - P + \tfrac{1}{2} e_q \dot{P}(\dot{P}+1) - \dot{P} \;, \tag{5.8}$$

where e_q equals 1, 2 or 4, depending on whether the Clifford algebra is of real, complex or quaternionic type (see Table 3). Hence $e_q = 1$ for $q = 0, 1, 7 \bmod 8$, $e_q = 2$ for $q = 2, 6 \bmod 8$ and $e_q = 4$ for $q = 3, 4, 5 \bmod 8$.

5.2 Symmetries of the real space.

The linear transformations of h^A that leave (5.2) invariant were already given in [17]. They constitute the isometries of the real special spaces and consist of (5.7) supplemented by (we rescaled some transformation parameters for future convenience)

$$\begin{aligned} \delta h^1 &= -2\lambda\, h^1 + 2\xi_i h^i \;, \\ \delta h^2 &= \lambda\, h^2 - \zeta^i\, h^i + \xi_\mu\, h^\mu \;, \\ \delta h^\mu &= \lambda\, h^\mu + \xi_\mu\, h^2 - \zeta^j\, \gamma_{\mu ij}\, h^i \;, \\ \delta h^i &= -\tfrac{1}{2}\lambda\, h^i + \xi_i\, h^2 - \zeta^i\, h^1 + \xi_j\, \gamma_{\mu ij}\, h^\mu + \tfrac{1}{2}\xi_\mu\, \gamma_{\mu ij}\, h^j \;. \end{aligned} \tag{5.9}$$

The symmetries corresponding to the parameters ζ^i can only exist when the tensor $\Gamma_{ijkl} = 0$; this tensor is defined by

$$\Gamma_{ijkl} \equiv \tfrac{3}{8} \Big[\gamma_{\mu(ij}\, \gamma_{kl)\mu} - \delta_{(ij}\, \delta_{kl)} \Big] \;, \tag{5.10}$$

and vanishes only when the corresponding Kähler and quaternionic spaces are symmetric[8]. When this is not the case the transformations (5.9) extend the number of symmetries of the previous subsection to

$$n - 1 + \tfrac{1}{2} q(q+1) + \tfrac{1}{2} e_q \Big(P(P+1) + \dot{P}(\dot{P}+1) \Big) - P - \dot{P} \;. \tag{5.11}$$

We now clarify the algebra $\mathcal{X}$ corresponding to these transformations. First it is easy to verify that generator $\underline{\lambda}$, associated with the infinitesimal transformation proportional to λ, extends the Cartan subalgebra of (5.6) to the Cartan subalgebra of $\mathcal{X}$. The rank of $\mathcal{X}$ is thus one unit higher than that of (5.6). Decomposing the isometry algebra with respect to $\underline{\lambda}$ we find

$$\mathcal{X} = \mathcal{X}_{-3/2} + \mathcal{X}_0 + \mathcal{X}_{3/2} \;, \tag{5.12}$$

[8]It does not vanish for $L(-1, P)$, where only the corresponding real space is symmetric, due to other isometries than those discussed here [14]. Henceforth we only refer to 'symmetric spaces' when all three varieties of the special spaces are symmetric.

where $\mathcal{X}_{-3/2}$ contains the generators associated with the parameters ζ^i (which is empty for the non-symmetric spaces), $\mathcal{X}_0$ consists of the generators associated with λ, ξ_μ, $A_{\mu\nu}$ and S_{ij}, while $\mathcal{X}_{3/2}$ contains the generators corresponding to the parameters ξ_i. We can simplify $\mathcal{X}_0$ to

$$\mathcal{X}_0 = so(1,1) \oplus so(q+1,1) \oplus \mathcal{S}_q(P,\dot{P}) \ , \tag{5.13}$$

where $so(1,1)$ corresponds to $\underline{\lambda}$ and the $so(q+1,1)$ algebra consists of the $so(q+1)$ algebra discussed in the previous subsection, combined with the generators associated with the infinitesimal transformations proportional to ξ_μ. It is convenient to decompose the fields h^A into h^1, h^M and h^i, where $M = 2$ or μ. The $so(q+1,1)$ infinitesimal transformations are then denoted by matrices $A^M{}_N$ with $\xi_\mu = A^2{}_\mu = A^\mu{}_2$ and $A^\mu{}_\nu = A_{\mu\nu}$, with $A_{MN} = \eta_{MP}\,A^P{}_N = -A_{NM}$, where the metric $\eta = \mathrm{diag}\,(-1,1,\ldots,1)$.

On the spinor fields h^i the group $SO(q+1,1)$ act as indicated in (5.9) (i.e. it transforms as a spinor in de Sitter space). From a more formal point of view, this can be understood as follows. The spinor fields do not allow a realization of $\mathcal{C}(q+1,1)$, but its subset consisting of the even elements, $\mathcal{C}^+(q+1,1)$, is always isomorphic with $\mathcal{C}(q+1,0)$, such that $SO(q+1,1)$ can act on h^i. To see this in more detail, let us first double the representation space and realize a (not necessarily irreducible) representation of the Clifford algebra $\mathcal{C}(q+1,1)$. This representation thus acts on $2\mathcal{D}_{q+1}$ coordinates, which we decompose into an equal number of components with upper and with lower indices, (ψ^i,ψ_i). As ψ^i and ψ_i transform in general according to inequivalent spinor representations of $SO(q+1,1)$, there is no invariant metric in spinor space that allows one to raise and lower spinorial indices, so that ψ^i and ψ_i remain independent. On this basis $\mathcal{C}(q+1,1)$ is realized by the gamma matrices

$$\gamma_M = \begin{pmatrix} 0 & \gamma_{M\,ik} \\ \gamma_M{}^{jl} & 0 \end{pmatrix} \ , \tag{5.14}$$

with $\gamma_{\mu\,ij} = \gamma_\mu{}^{ij}$ equal to the gamma matrices introduced above, and $\gamma_2{}^{ij} = -\gamma_{2\,ij} = \delta_{ij}$. The generators of $SO(q+1,1)$ in the spinor representation are

$$\gamma_{MN}{}^i{}_j = \gamma_{[M}{}^{ik}\,\gamma_{N]\,kj} \ , \tag{5.15}$$

so that

$$\gamma_{\mu\nu}{}^i{}_j = \gamma_{[\mu}{}^{ik}\,\gamma_{\nu]kj} \ ; \qquad \gamma_{2\mu}{}^i{}_j = -\gamma_{\mu 2}{}^i{}_j = \gamma_\mu{}^{ij} \ . \tag{5.16}$$

With this notation the only non-zero components of the tensor d_{ABC} are

$$d_{1MN} = -\eta_{MN} \ ; \qquad d_{Mij} = \gamma_{Mij} \ , \tag{5.17}$$

where the coordinates h^i are assigned upper indices. Assigning upper indices to the generators $\underline{\xi}^i$ and lower indices to $\underline{\zeta}_i$ (so that the corresponding parameters are written as ξ_i and ζ^i, respectively), the commutators with the $so(q+1,1)$ generators are

$$[\underline{A}_{MN},\underline{\xi}^i] = -\tfrac{1}{2}\gamma_{MN}{}^i{}_j\,\underline{\xi}^j \ ; \qquad [\underline{A}_{MN},\underline{\zeta}_i] = \tfrac{1}{2}\underline{\zeta}_j\,\gamma_{MN}{}^j{}_i \ , \tag{5.18}$$

and the combined transformations of (5.7) and (5.9) are incorporated in the matrix ($A = (1,M,i)$ and $B = (1,N,j)$)

$$\tilde{B}^A{}_B = \begin{pmatrix} -2\lambda & 0 & 2\xi_j \\ 0 & \lambda\,\delta^M_N + A^M{}_N & -\zeta^k\,\gamma^M{}_{kj} \\ -\zeta^i & \gamma_N{}^{ik}\xi_k & -\tfrac{1}{2}\lambda\delta^i_j + \tfrac{1}{4}A^{PQ}\,(\gamma_{PQ})^i{}_j + S^i{}_j \end{pmatrix} \ . \tag{5.19}$$

To find the isotropy group, consider points with $h^\mu = h^i = 0$. Then h^1 depends on h^2 because of (2.24), so that one is left with a one-dimensional subspace. For these points it is easy to show that the metric (defined in (2.23)) is negative definite as required. None of these points is left invariant by the symmetries (5.9) (this is not so for the symmetric spaces where the isotropy group contains transformations associated with a linear combination of ξ_i and ζ^i), so that the (compact) isotropy group equals

$$H = SO(q+1) \otimes \mathcal{S}_q(P,\dot{P}) \ . \tag{5.20}$$

The $(n-1)$-dimensional solvable subalgebra of the non-compact isometry group G consists of two parts, namely the r generators contained in $\mathcal{X}_{3/2}$, and the $q+2$ generators belonging to the solvable subalgebra of $\mathcal{X}_0$; those are the generator $\underline{\lambda}$ and the $q+1$ generators belonging to the solvable subalgebra of $so(q+1,1)$ (see appendix A). The rank of the $so(q+1,1)$ equals 1 (for $q \geq 0$; for $q=-1$ the algebra is empty, so that the rank is 0), so that the rank of the full solvable algebra equals 2 (or 1 for $q=-1$). Obviously the real homogeneous spaces have the form of $\frac{SO(q+1,1)}{SO(q+1)}$ supplemented with $r+1$ coordinates associated with the rigid motions parametrized by λ and ξ_i, which act as translations and transform linearly under $SO(q+1,1)\otimes\mathcal{S}_q(P,\dot{P})$. For symmetric spaces the situation is different as one has to take into account the extra isometries contained in $\mathcal{X}_{-3/2}$.

Note that for $q=-1$ we have $\mathcal{X}_0 = so(1,1)\oplus so(P)$. Then the $r=P$ generators of $\mathcal{X}_{3/2}$ and the additional isometries found in [14] extend this algebra to $so(P+1,1)$, and one finds a symmetric space as indicated in Table 2.

For $q=0$ we have $\mathcal{X}_0 = so(1,1)\oplus so(1,1)\oplus so(P)\oplus so(\dot{P})$. The generators $\underline{\lambda}$ and $\underline{\alpha}$ (the one corresponding to ξ_μ), can be recombined such that one linear combination commutes with the P generators in $\mathcal{X}_{3/2}$ that transform under $SO(P)$, and the other linear combination commutes with the remaining $\dot{P}$ ones transforming under $SO(\dot{P})$. The space then becomes a local product of a $(P+1)$- and a $(\dot{P}+1)$-dimensional space (each of rank 1). The isometry algebra is a contraction of $so(P+1,1)\oplus so(\dot{P}+1,1)$. These spaces were denoted by $Y(P,\dot{P})$ in Table 2. If $q=\dot{P}=0$ the space is symmetric and one must include the P generators contained in $\mathcal{X}_{-3/2}$. This then leads to the isometry algebra $so(P+1,1)\oplus so(1,1)$, as exhibited in Table 2.

5.3 Symmetries of the Kähler space.

The special Kähler spaces are based on the function $F(X)$, which for the 'very special' spaces takes the form

$$F(X) = \frac{3i}{X^0}\left\{-\eta_{MN}\,X^M X^N X^1 + \gamma_{M\,ij}\,X^M X^i X^j\right\} \ . \tag{5.21}$$

The previous isometry algebra is now extended with the transformations corresponding to β, b^A and the solutions of (4.22). So we first have to solve the latter. As the spaces are homogeneous, we may do so at any point in the domain. It is convenient to consider points in the subspace $x^\mu = x^i = 0$, so that only x^1 and x^2 are non-vanishing (and possibly the real parts of z^A, but they never appear). The metric is diagonal in this parametrization and is given by

$$g_{11} = -\left(x^1\right)^{-2} \ ; \quad g_{22} = -2\left(x^2\right)^{-2} \ ; \quad g_{\mu\nu} = -2\delta_{\mu\nu}\left(x^2\right)^{-2} \ ; \quad g_{ij} = -2\delta_{ij}\left(x^1x^2\right)^{-1} \ . \tag{5.22}$$

As $dxxx = 3x^1\,(x^2)^2$, the domain of positivity is x^1, $x^2 > 0$. The non-zero components of C^{ABC} are (cf. (2.16))

$$C^{122} = \tfrac{3}{4}\ ; \qquad C^{1\mu\nu} = -\tfrac{3}{4}\delta^{\mu\nu}\ ; \qquad C^{2ij} = -\tfrac{3}{8}\delta^{ij}\ ; \qquad C^{\mu ij} = \tfrac{3}{8}\gamma_{\mu ij} \tag{5.23}$$

or, in $so(q+1,1)$-covariant notation,

$$C^{1MN} = -\tfrac{3}{4}\eta^{MN}\ ; \qquad C^{Mij} = \tfrac{3}{8}\gamma^{Mij}\ . \tag{5.24}$$

The curvature tensor, which appears in the commutation rules of the isometries, follows directly from the above result. Its non-zero components in the points $x^\mu = x^i = 0$ are

$$\begin{aligned}
&R^1{}_{11}{}^1 = -2\ ; \qquad R^1{}_{1j}{}^i = -\delta^i_j\ ,\\
&R^1{}_{ij}{}^M = -\gamma^M{}_{ij}\ ; \qquad R^i{}_{1M}{}^j = -\tfrac{1}{2}\gamma_M{}^{ij}\ ,\\
&R^P{}_{MN}{}^Q = -2\delta_M^{(P}\,\delta_N^{Q)} + \eta_{MN}\,\eta^{PQ}\ ,\\
&R^N{}_{Mi}{}^j = -\tfrac{1}{2}\gamma_M{}^{jk}\,\gamma^N{}_{ki}\ ,\\
&R^k{}_{ij}{}^l = -2\delta_i^{(k}\,\delta_j^{l)} + \tfrac{1}{2}\gamma_{Mij}\,\gamma^{Mkl}\ .
\end{aligned} \tag{5.25}$$

Furthermore, a straightforward calculation gives for the non-zero components of E^E_{ABCD},

$$E^1_{ijkl} = 2\Gamma_{ijkl}\ ; \qquad E^i_{Mjkl} = \Gamma_{jklm}\gamma_M{}^{mi}\ , \tag{5.26}$$

where Γ_{ijkl} was defined in (5.10). We consider the non-symmetric spaces where this tensor is non-zero. Already in [17] we concluded that there are no non-trivial solutions of $\Gamma_{ijkl}\,a_l = 0$ in that case. Therefore there are only hidden Kähler symmetries associated with the parameters a_μ and a_2.

According to (2.26) the isometry algebra $\mathcal{W}$ can be decomposed into three eigenspaces of $\underline{\beta}$; $\mathcal{W}_{-2/3}$ contains the transformations corresponding to a_M, $\mathcal{W}_0 = \mathcal{X} \oplus \underline{\beta}$, and $\mathcal{W}_{2/3}$ contains the generators associated with the parameters b^A. The total number of isometries is now

$$2(n+1) + \tfrac{1}{2}q(q+3) + \tfrac{1}{2}e_q\left(P(P+1) + \dot{P}(\dot{P}+1)\right) - P - \dot{P}\ . \tag{5.27}$$

The parameters b^A decompose with respect to $SO(q+1,1)$ into b^1, b^M and b^i. Most of the commutation rules are indicated in Table 4. The $\mathcal{S}_q$ representation denoted by v denotes the vector (or defining) representation. The $so(q+1,1)$ representation denoted by v, s and $\bar{s}$ denotes the vector, spinor and conjugate spinor representation, respectively. Hence we have

$$\begin{aligned}
[\underline{A}_{MN},\,\underline{a}^P] &= 2\underline{a}_{[M}\,\delta^P_{N]}\ ,\\
[\underline{A}_{MN},\,\underline{\xi}^i] &= -\tfrac{1}{2}\gamma_{MN}{}^i{}_j\,\underline{\xi}^j\ ,\\
[\underline{A}_{MN},\,\underline{b}_i] &= \tfrac{1}{2}\underline{b}_j\,\gamma_{MN}{}^j{}_i\ .
\end{aligned} \tag{5.28}$$

To calculate the last commutator of (4.23) we can use the curvature tensor as determined above for points in the two-dimensional complex subspace, because we know that the components $R^M{}_{AB}{}^C$ are related to the hidden symmetries parametrized by a_M, and therefore constant. In this way we obtain

$$\begin{aligned}
[\underline{\xi}^j,\,\underline{b}_M] &= -\underline{b}_i\,\gamma_M{}^{ij}\ ,\\
[\underline{\xi}^j,\,\underline{b}_i] &= -2\delta_i^j\,\underline{b}_1\ ,\\
[\underline{b}_M,\,\underline{a}^N] &= \delta^N_M(\underline{\beta} - \tfrac{1}{3}\underline{\lambda}) - \underline{A}^N{}_M\ ,\\
[\underline{b}_i,\,\underline{a}^N] &= -\tfrac{1}{2}\gamma^N{}_{ij}\,\underline{\xi}^j\ .
\end{aligned} \tag{5.29}$$

Table 4: Roots of the isometries of the non-symmetric homogeneous very special Kähler spaces.

generator	$\mathcal{S}_q$	$so(q+1,1)$	$\underline{\lambda}$	$\underline{\beta}$	$\underline{\beta}-\frac{1}{3}\underline{\lambda}$	$\frac{2}{3}\underline{\lambda}+\underline{\beta}$
$\underline{b}_1$	0	0	2	$\frac{2}{3}$	0	2
$\underline{\xi}^i$	v	s	$\frac{3}{2}$	0	$-\frac{1}{2}$	1
$\underline{b}_i$	v	$\bar{s}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	1
$\underline{a}^M$	0	v	1	$-\frac{2}{3}$	-1	0
$\underline{b}_M$	0	v	-1	$\frac{2}{3}$	1	0

Note that $\underline{b}_1$ commutes with the generators discussed above, as well as with the $so(q+1,1)$ generators. Furthermore, it is easy to verify that the generators $\underline{A}^M{}_N$, $\underline{a}^M$, $\underline{b}_M$ and $\underline{\beta}-\frac{1}{3}\underline{\lambda}$ define the algebra $so(q+2,2)$. Extending the indices M with two more values, which we indicate by a and b, we may define $\underline{A}_{aM}\equiv\underline{a}_M$, $\underline{A}_{bM}\equiv\underline{b}_M$, and $\underline{A}_{ab}=-\underline{A}_{ba}=\underline{\beta}-\frac{1}{3}\underline{\lambda}$. The invariant metric η_{MN} is then extended to an $SO(q+2,2)$ invariant metric by including $\eta_{ab}=\eta_{ba}=-1$.

According to the above commutation relations, the spinor representation of $SO(q+2,2)$ acts on the generators $(\underline{\xi}^i,\underline{b}_i)$. This can be understood from the equivalence relations $\mathcal{C}^+(q+2,2)\simeq\mathcal{C}(q+2,1)\simeq\mathcal{C}(q+1,0)\otimes\mathrm{I\!R}(2)$.

The linear combination of $\underline{\lambda}$ and $\underline{\beta}$ that commutes with $so(q+2,2)$ is $\underline{\lambda}'=\frac{2}{3}\underline{\lambda}+\underline{\beta}$. The grading of the full isometry algebra with respect to $\underline{\lambda}'$ is as follows,

$$\begin{aligned}
&\mathcal{W}=\mathcal{W}_0'\oplus\mathcal{W}_1'\oplus\mathcal{W}_2'\ ,\\
&\mathcal{W}_0'=\underline{\lambda}'\oplus so(q+2,2)\oplus\mathcal{S}_q(P,\dot{P})\ ,\\
&\mathcal{W}_1'=\underline{\xi}^i\oplus\underline{b}_i=(1,s,v)\ ,\\
&\mathcal{W}_2'=\underline{b}_1=(2,0,0)\ ,
\end{aligned}\tag{5.30}$$

where, for $\mathcal{W}_1'$ and $\mathcal{W}_2'$, we indicated the representations with respect to the three subalgebras of $\mathcal{W}_0'$. Note that because of the equivalence of the Clifford algebras indicated above, $\mathcal{S}_q(P,\dot{P})$ is also the centralizer of the $\mathcal{C}^+(q+2,2)$ in the relevant representation. In this picture, the generators of grading 0 form again a reductive algebra, while the other generators have positive grading : at grading 1 the generators constitute a direct product of a spinor representation of $so(q+2,2)$ with a vector representation of $\mathcal{S}_q$; at grading 2 there is just a singlet generator. No generators with negative grading occur. This is different for the symmetric spaces, where we have additional isometries associated with the parameters ζ^i, a_1 and a_i. They decompose into generators $(\underline{\zeta}_i,\underline{a}^i)$ at grading -1 and $\underline{a}^1$ at grading -2, so that the isometry algebra is semisimple.

Let us briefly consider the simplest cases. For the homogeneous spaces related to $L(-1,P)$, the reductive algebra $\mathcal{W}_0'$ is $\underline{\lambda}'\oplus so(2,1)\oplus so(P)$. The generators in $\mathcal{W}_1'$ form a $(1,2,P)$ representation of this algebra. For $q=0$, we have the so-called $K(P,\dot{P})$ spaces and

$$\mathcal{W}_0' \;=\; \underline{\lambda}'\oplus so(2,2)\oplus so(P)\oplus so(\dot{P})$$

$$= \underline{\lambda}' \oplus (so(2,1) \oplus so(P)) \oplus (so(2,1) \oplus so(\dot{P})) \ . \tag{5.31}$$

The generators in $\mathcal{W}_1'$ decompose into two separate representations, $(1,2,P,0,0)$ and $(1,0,0,2,\dot{P})$. Because of the presence of $\underline{\lambda}'$ and $\underline{b}_1$ in the isometry algebra, this does not lead to a factorization of the corresponding Kähler space. The latter two generators form a so-called canonical subalgebra, characteristic for the Kählerian algebras [21, 16].

From the transformation rules (2.22), it is obvious that the coordinates z^A do not transform linearly under the isometry group. For the real spaces, discussed in the previous subsection, the transformations were realized linearly on the fields h^A, which are, however, subject to the constraint (2.24), and the cubic polynomial (5.2) is invariant under the isometry transformations. For the Kähler spaces there are the corresponding fields X^I, which do not transform linearly either as is shown in (2.10). Furthermore the homogeneous function $F(X)$ given in (5.21) is not invariant under the full isometry group, and neither is it invariant under the isotropy group. The latter group coincides with the compact subgroup associated with $\mathcal{W}_0'$ and equals

$$H = SO(q+2) \otimes SO(2) \otimes \mathcal{S}_q(P,\dot{P}) \ . \tag{5.32}$$

The $SO(q+2)$ group is associated with the compact subgroup of $SO(q+1,1)$ and linear combinations of the generators $\underline{a}^\mu$ and $\underline{b}_\mu$; the $SO(2)$ group is generated by a linear combination of $\underline{a}^2$ and $\underline{b}_2$ (and not by linear combinations of $\underline{\lambda}$ and $\underline{\beta}$). In general it is not possible to use a symplectic reparametrization (cf. appendix C) to bring $F(X)$ into a different form that is manifestly invariant under the isotropy group H.

5.4 Symmetries of the quaternionic space.

The symmetries of the generic quaternionic very special spaces are summarized by table 1. In the previous subsection, we have already identified the symmetries corresponding to $\tilde{B}^A{}_B$ and a_A. As we know that the condition for the existence of the a_A symmetries coincides with the condition for the existence of the $\hat{\beta}_A$ symmetries, there are thus $q+2$ solutions associated with the parameters $\hat{\beta}_M$. We are left with the $\hat{\alpha}^A$ symmetries, whose existence is governed by condition (4.45). From (4.34) we derive at once, using (5.23), that there is a symmetry associated with $\hat{\alpha}^1$. Using the E coefficients in (5.26) establishes that the non-symmetric spaces have no other symmetries. Another approach is to consider (5.17) and to note that additional symmetries associated with $\hat{\alpha}^A$ with $A \neq 1$, would lead to $\hat{\beta}_A$ symmetries of the form $\hat{\beta}_A = d_{ABC}\,\hat{\alpha}^B$ that are not realized.

Hence the isometry algebra of the homogeneous quaternionic spaces consist of the algebra (5.30) of the corresponding Kähler spaces extended with $\hat{\beta}^M$ and $\hat{\alpha}_1$. Using (3.8), (4.33) and (5.19), we can group the generators into representations of $\underline{\epsilon}_0 \oplus \underline{\lambda}' \oplus so(q+2,2) \oplus \mathcal{S}_q(P,\dot{P})$. The results are listed in Table 5, where, in the first column, the generators that constitute an $so(q+2,2)$ representation are ordered according to their weight under the $so(q+2,2)$ generator $\underline{\beta} - \frac{1}{3}\underline{\lambda}$. For the vector representation these weights are $(-1,0,1)$, and for the spinor representation $(-\frac{1}{2},\frac{1}{2})$.

From the table we see immediately that $so(q+2,2)$ can be extended with $(\underline{\hat{\alpha}}_1, \underline{\hat{\beta}}^M, \underline{\hat{\beta}}^0)$, $(\underline{\alpha}_0, \underline{\alpha}_M, \underline{\beta}^1)$ and $\underline{\lambda}' - 2\underline{\epsilon}_0$ to $so(q+3,3)$. This can be checked by combining (3.9) and (4.36). The remaining generators can be grouped into $so(q+3,3)$ representations. Denoting the linear combination of $\underline{\epsilon}_0$ and $\underline{\lambda}'$ that commutes with $so(q+3,3)$, by $\underline{\epsilon}' \equiv 2\underline{\epsilon}_0 + \underline{\lambda}'$, we find

Table 5: Roots of the isometries of the non-symmetric homogeneous very special quaternionic spaces

generator	$\mathcal{S}_q$	$so(q+2,2)$	$\underline{\lambda}'$	$\underline{\epsilon}_0$	$\underline{\lambda}'-2\underline{\epsilon}_0$	$\underline{\lambda}'+2\underline{\epsilon}_0$
$\underline{\epsilon}_+$	0	0	0	1	-2	2
$(\underline{\alpha}_1,\underline{\beta}^M,\underline{\beta}^0)$	0	v	1	$\frac{1}{2}$	0	2
$\underline{b}_1$	0	0	2	0	2	2
$(\underline{\alpha}_i,\underline{\beta}^i)$	v	s	0	$\frac{1}{2}$	-1	1
$(\underline{\xi}^i,\underline{b}_i)$	v	$\bar{s}$	1	0	1	1
$(\underline{\hat{\alpha}}_1,\underline{\hat{\beta}}^M,\underline{\hat{\beta}}^0)$	0	v	1	$-\frac{1}{2}$	2	0
$(\underline{\alpha}_0,\underline{\alpha}_M,\underline{\beta}^1)$	0	v	-1	$\frac{1}{2}$	-2	0

a decomposition of the isometry algebra as in the previous subsection :

$$\begin{aligned} \mathcal{V} &= \mathcal{V}_0'+\mathcal{V}_1'+\mathcal{V}_2' \ , \\ \mathcal{V}_0' &= \underline{\epsilon}'\oplus so(q+3,3)\oplus\mathcal{S}_q(P,\dot{P}) \ , \\ \mathcal{V}_1' &= (\underline{\xi}^i,\underline{b}_i)\oplus(\underline{\alpha}_i,\underline{\beta}^i)=(1,s,v) \ , \\ \mathcal{V}_2' &= \underline{\epsilon}_+\oplus(\underline{\alpha}_1,\underline{\beta}^M,\underline{\beta}^0)\oplus\underline{b}_1=(2,v,0) \ , \end{aligned} \tag{5.33}$$

where we indicated the representation of $\mathcal{V}_1'$ and $\mathcal{V}_2'$ according to the three subalgebras of $\mathcal{V}_0'$. The above result fully determines the generic structure of the isometry algebra for the (non-symmetric) homogeneous quaternionic spaces. The isotropy group is the maximal compact subgroup of the isometry group, which equals

$$H=SO(q+3)\otimes SO(3)\otimes\mathcal{S}_q(P,\dot{P}) \ . \tag{5.34}$$

For the symmetric spaces the isometry algebra is extended with additional generators, with negative eigenvalues of $\underline{\epsilon}'$,

$$\begin{aligned} \mathcal{V}_{-1}' &= (\underline{a}^i,\underline{\zeta}_i)\oplus(\underline{\hat{\alpha}}_i,\underline{\hat{\beta}}^i)=(-1,s,v) \ , \\ \mathcal{V}_2' &= \underline{\epsilon}_-\oplus(\underline{\hat{\alpha}}_0,\underline{\hat{\alpha}}_M,\underline{\hat{\beta}}^1)\oplus\underline{a}^1=(-2,v,0) \ . \end{aligned} \tag{5.35}$$

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A Normal spaces

For homogeneous spaces the isometries act transitively on the manifold so that every two points are related by an element of the isometry group. The orbit swept out by the action of the isometry group G from any given point is (locally) isomorphic to the coset space G/K, where K is the isotropy group of that point. For non-compact homogeneous spaces where K is the maximal compact subgroup of G, there exists a solvable subgroup that acts transitively and whose dimension is equal to the dimension of the space. Such spaces are called *normal*. This solvable algebra s determines the coset completely, i.e. $\frac{G}{K} = e^s$. We will show how to obtain s for any coset.

The construction of the solvable algebra follows from the Iwasawa decomposition of the group,

$$G = K\,A\,N\ , \tag{A.1}$$

where K is the maximal compact subgroup; the remaining factor is then decomposed into its maximal Abelian subgroup A and a nilpotent group N. The solvable subgroup is $F = A\,N$.

For normal spaces the compact subgroup can be divided out according to the above decomposition so that one is left with the group space associated with F. To see how this works, we first consider the example of $SO(n,1)/SO(n)$, and show how this coset is indeed the exponential of a solvable algebra. Then we explain the Iwasawa decomposition for the algebra. Finally we illustrate how this decomposition works for the example of $so(n,1)$.

One decomposes the generators of $SO(n,1)$ using an off-diagonal metric decomposed according to $(n+1) = 1 \oplus (n-1) \oplus 1$,

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{1} & 0 \\ 1 & 0 & 0 \end{pmatrix}\ . \tag{A.2}$$

The generators of $SO(n,1)$ are subject to the condition $t^{\mathrm{T}} = -\eta\, t\, \eta$ and can be decomposed as follows,

$$t \propto \begin{pmatrix} \alpha & -\xi^{\mathrm{T}} & 0 \\ \zeta & so(n-1) & \xi \\ 0 & -\zeta^{\mathrm{T}} & -\alpha \end{pmatrix}\ . \tag{A.3}$$

Here ξ and ζ are $(n-1)$-dimensional real vectors. The $so(n-1)$ generators together with the remaining $n-1$ compact generators characterized by $\xi = \zeta$ define the maximal compact subalgebra $so(n)$. Observe that the vector left invariant under $SO(n)$ is equal to $(1, 0, \ldots 0, -1)$. Acting on this vector with $SO(n,1)$ generates an orbit isomorphic to $SO(n,1)/SO(n)$ as well as to $F = \exp s$, where s is the solvable algebra associated with the generators parametrized by α and ξ. It is not difficult to give this group explicitly in terms of these parameters,

$$\mathrm{e}^{s(\alpha,\xi)} = \begin{pmatrix} \mathrm{e}^{\alpha} & \dfrac{1-\mathrm{e}^{\alpha}}{\alpha}\,\xi^{\mathrm{T}} & \dfrac{1-\cosh\alpha}{\alpha^2}\,\xi^2 \\ 0 & 0 & \dfrac{1-\mathrm{e}^{-\alpha}}{\alpha}\,\xi \\ 0 & 0 & \mathrm{e}^{-\alpha} \end{pmatrix} \tag{A.4}$$

The rank of the coset is defined as the rank of the solvable algebra, which in this case is equal to 1. The Cartan subalgebra consists of the generator associated with the parameter α.

We now present the general procedure (see e.g. [29]) for the Iwasawa decomposition of a non-compact algebra

$$g = k \oplus p \ , \tag{A.5}$$

such that

$$u = k \oplus ip \tag{A.6}$$

is compact. We will thus explain how to find the solvable algebra s. First, choose a maximal number of commuting elements from p, and denote them by h_p. This set can be extended to a Cartan subalgebra (CSA) h by including suitable elements from k. We order the elements in the CSA by putting the elements of h_p in front, and then define the set of positive roots Δ^+. Now n is the subset of Δ^+ defined by those elements which are not completely in k. (More mathematically : define an automorphism θ such that $\theta(k) = k$ and $\theta(p) = -p$, then n is the subset of Δ^+ consisting of elements x such that $\theta(x) \neq x$). Equivalently, n consists of the roots which are strictly positive with respect to h_p only. Then

$$s = h_p \oplus n \tag{A.7}$$

is a solvable algebra, which has the same dimension as p (see [29]) and e^s is isomorphic with the coset g/k. The dimension of the Cartan subalgebra of s equals the *rank* of the homogeneous space.

Consider now the example of $so(n,1)$. Let us define the algebra by using the vector representation. Indices $M, N, \ldots,$ run over the values $0, 1, \ldots, n$. Indices μ, ν run from 1 to n. Furthermore we will also use indices $m, n, \ldots$ running from 2 to n. We define transformations

$$\delta z^M = A^M{}_N z^N \ , \tag{A.8}$$

and raise or lower indices with the metric $\eta^{MN} = \mathrm{diag}\,(-1, 1, \ldots, 1)$. Then A_{MN} is antisymmetric. The generators $\underline{A}$ are defined by $\delta = \frac{1}{2} A^M{}_N \underline{A}^N{}_M$. The compact ones $(\underline{A}^\mu{}_\nu)$ are antisymmetric operators, while the non-compact ones $(\underline{A}^0{}_\mu)$ are symmetric. Their action is

$$\underline{A}_{MN} z^N = \delta^P_N z_M - \delta^P_M z_N \ . \tag{A.9}$$

The commutation relations

$$[\underline{A}_{MN}, \underline{A}_{PQ}] = \underline{A}_{MQ}\, \eta_{NP} - \underline{A}_{NQ}\, \eta_{MP} - \underline{A}_{MP}\, \eta_{NQ} + \underline{A}_{NP}\, \eta_{MQ} \tag{A.10}$$

imply that there are no mutually commuting non-compact generators. The space is therefore of rank 1, and we can choose $\underline{\alpha} \equiv \underline{A}^0{}_1$ as the generator in h_p (the full h includes also the Cartan subalgebra of $so(n-1)$). Now we have to diagonalize the commutator of h_p with the other generators. This requires the linear combinations

$$\underline{A}^\pm{}_m = \frac{1}{\sqrt{2}} \left(\underline{A}^0{}_m \pm \underline{A}^1{}_m \right) \ . \tag{A.11}$$

On this basis the metric becomes equal to the metric (A.2). The generators $\underline{A}^+{}_m$ have a positive root with respect to $\underline{\alpha}$, and correspond to the parameters ξ_m in (A.3). Thus we find that the $n-1$ generators $\underline{\xi}_m$ and $\underline{\alpha}$ generate the n-dimensional coset $so(n,1)/so(n)$; the only non-zero commutators of the solvable algebra are

$$[\underline{\alpha},\,\underline{\xi}_m]=\underline{\xi}_m\ . \tag{A.12}$$

B Useful formulae

We collect here several formulae which are useful in calculations. The matrices in (2.2) have simple contractions with the vector X

$$\begin{aligned}
X^I\mathcal{M}_{I\bar{J}} &= 0\\
X^I\mathcal{N}_{IJ} &= -\frac{1}{4}X^IF_{IJ}=-\frac{1}{4}F_J\\
(XN\bar{X})(XN)_I &= -(XNX)\bar{X}^J(\mathcal{N}+\bar{\mathcal{N}})_{JI}.
\end{aligned} \tag{B.1}$$

For the inverse matrices there are the following relations (the matrix $\mathcal{M}^{-1\,\bar{A}B}$ is the inverse of $\mathcal{M}_{A\bar{B}}$ as $n\times n$ matrix)

$$\begin{aligned}
&\mathcal{M}^{-1\,\bar{A}B}=(zN\bar{z})^{-1}g^{\bar{A}B}\\
&\qquad\quad=(N^{-1})^{\bar{A}B}-(N^{-1})^{\bar{A}0}\,z^B-\bar{z}^{\bar{A}}\,(N^{-1})^{0B}+(N^{-1})^{00}\,\bar{z}^{\bar{A}}\,z^B\\
&(N^{-1})^{IJ}=(\mathcal{N}+\bar{\mathcal{N}})^{-1\ IJ}+\frac{X^I\bar{X}^J+\bar{X}^IX^J}{XN\bar{X}}\\
&(\mathcal{N}+\bar{\mathcal{N}})^{-1\ IK}N_{KJ}=(N^{-1})^{IK}\mathcal{M}_{K\bar{J}}-\frac{X^I(N\bar{X})_J}{XN\bar{X}}\\
&\mathcal{N}\frac{1}{\mathcal{N}+\bar{\mathcal{N}}}\bar{\mathcal{N}}=\bar{\mathcal{N}}\frac{1}{\mathcal{N}+\bar{\mathcal{N}}}\mathcal{N}=\mathcal{N}-\mathcal{N}\frac{1}{\mathcal{N}+\bar{\mathcal{N}}}\mathcal{N}
\end{aligned} \tag{B.2}$$

We have defined in [15] also $\left(N^{-1}\right)^{00}=\Delta^{-1}$ and the matrix $\left(n^{-1}\right)^{AB}$ as the inverse of N_{AB}. We then have

$$\Delta=zN\bar{z}-zN\left(n^{-1}\right)N\bar{z}=N_{00}-N_{0A}\left(n^{-1}\right)^{AB}N_{B0}\ . \tag{B.3}$$

and[9]

$$\mathcal{M}^{-1\,A\bar{B}}=\left(n^{-1}\right)^{AB}+\Delta^{-1}\left(n^{-1}Nz\right)^{A}\left(n^{-1}N\bar{z}\right)^{B} \tag{B.4}$$

Furthermore we have the following result: if $X^IS_I=0$ then

$$\begin{aligned}
R_I\left(N^{-1}\right)^{IJ}S_J &= R_A\left(n^{-1}\right)^{AB}S_B\\
&\quad+\Delta^{-1}\left(-R_0+R_A(n^{-1}N)^A_{\ 0}\right)\left(n^{-1}Nz\right)^BS_B\ .\\
&= R_A\mathcal{M}^{-1\,\bar{A}B}S_B-\Delta^{-1}R_I\bar{z}^I\left(n^{-1}Nz\right)^BS_B\ .
\end{aligned} \tag{B.5}$$

The last term disappears if moreover R_I satisfies $\bar{X}^IR_I=0$, a result which was given already in [15].

[9] The implicit contractions over indices are always over the maximal set of indices for the matrices : $\left(n^{-1}Nz\right)^A\equiv\left(n^{-1}\right)^{AB}N_{BI}z^I$

C Symplectic reparametrizations of special Kähler manifolds

In [3] it was noted that two different F functions can give rise to equivalent theories (provided the vector fields are abelian). As an example it was demonstrated that two $n=1$ theories discussed in [15] with $F^{\rm I} = i(X^1)^3/X^0$ and $F^{\rm II} = (X^0)^{1/2}\,(X^1)^{3/2}$ lead to equivalent field equations. The relation between two functions that describe equivalent theories takes the form of symplectic reparametrizations of the same type as the duality invariances discussed in [5]; the invariances can be viewed as a special subclass. The space of theories describing n abelian vector multiplets coupled to $N=2$ supergravity is thus parametrized by holomorphic functions of $n+1$ variables that are homogeneous of second degree, divided by $Sp(2n+2,\mathbb{R})$ transformations [3]. In addition, functions that differ by a quadratic polynomial with imaginary coefficients, are equivalent [5]. The symplectic reparametrizations are relevant in the treatment of Calabi-Yau manifolds and related phenomena and for bringing the theory in a form where certain subgroups of the duality invariances are linearly realized (see, for instance, [8, 10, 16, 30, 31]). Here we summarize their main features and clarify a number of related issues.

For the scalars the possibility of such reparametrizations is already suggested by the part of the action depending only on scalars and gravitons (after elimination of the auxiliary D field, see [22]), which is invariant under local scale and phase transformations,

$$\begin{aligned} 4e^{-1}\mathcal{L}_{(0,2)} &= \tfrac{1}{6}R\,(\bar X^I\,F_I + X^I\,\bar F_I) \\ &\quad +(\partial_\mu - iA_\mu)\bar X^I\,(\partial^\mu + iA^\mu)F_I + (\partial_\mu + iA_\mu)X^I\,(\partial^\mu - iA^\mu)\bar F_I, \end{aligned} \tag{C.1}$$

where A_μ is the (auxiliary) gauge field associated with the phase transformations. One may now perform linear redefinitions of (X^I, F_I), which commute with the local scale and phase transformations by virtue of the fact that $F(X)$ is holomorphic and homogeneous of second degree,

$$\begin{aligned} \tilde X^I &= U^I{}_J\,X^J - \tfrac{1}{2}iZ^{IJ}\,F_J, \\ \tilde F_I &= V_I{}^J\,F_J + 2iW_{IJ}\,X^J, \end{aligned} \tag{C.2}$$

where U, V, W and Z are constant $(n+1)\times(n+1)$ matrices. The above redefinition preserves the form of the Lagrangian (C.1) provided that (for generic functions $F(X)$, so that X^I and F_I may be considered as linearly independent functions) the matrix

$$\mathcal{O} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \tag{C.3}$$

satisfies

$$\mathcal{O}^{-1} = \Omega\,\mathcal{O}^\dagger\,\Omega^{-1}\,, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{C.4}$$

For functions F for which F_I and X^I are dependent, the derivation given above is not fully applicable. In that case it is best to proceed to the subsequent derivation for the spinor and vector fields, which is more general, and subsequently return to the scalar-field sector [5, 10].

We shall need that $\partial\tilde{X}^I/\partial X^J$ is non-singular. In other words, that the first line of (C.2), with F_I a function of X, defines an invertable relation between $\tilde{X}^I$ and X^I. This is an additional non-trivial condition on the matrix (C.3), which depends on F. To see this, consider for instance the transformation defined by $\tilde{X}^1 = \frac{1}{2}iF_1$ and $\tilde{F}_1 = 2iX^1$, with all other variables left unchanged, which satisfies (C.4). However, if F depends linearly on X^1, this transformation is not allowed, as $\tilde{X}^I$ does not depend on X^1, so that the relation between $\tilde{X}^I$ and X^I is not invertable.

In order to remain within the same class of Lagrangians, one must also require that $\tilde{F}_I$ can be written as the derivative of some new function $\tilde{F}(\tilde{X})$,

$$\tilde{F}_I = \frac{\partial \tilde{F}(\tilde{X})}{\partial \tilde{X}^I} \ , \tag{C.5}$$

which is again holomorphic and homogeneous of second degree. This is the case when

$$\tilde{F}_{IJ} \equiv \frac{\partial \tilde{F}_I}{\partial \tilde{X}^J} = \Big(V_I{}^K F_{KL} + 2iW_{IL}\Big) \frac{\partial X^L}{\partial \tilde{X}^J} \tag{C.6}$$

is a symmetric matrix. Here we made use of the fact that the transformation $\tilde{X}^I(X^I)$ is invertable, as explained above. Using the first line of (C.2) this integrability condition requires that

$$2i(U^{\rm T}W)_{IJ} + (U^{\rm T}V)_I{}^K\, F_{KJ} + F_{IK}\,(Z^{\rm T}W)^K{}_J - \tfrac{i}{2}F_{IK}\,(Z^{\rm T}V)^{KL}\,F_{LJ} \tag{C.7}$$

be symmetric in I and J. For a general function F the above condition implies that the first and the last term are separately symmetric; furthermore we assume that the identity is the only constant matrix that commutes with F_{IJ}. As a result one finds

$$\mathcal{O}^{-1} = \Omega\, \mathcal{O}^{\rm T}\, \Omega^{-1} \ . \tag{C.8}$$

Note, however, that for special functions this restriction may not necessarily be required at this point.

Combining (C.8) with (C.4) we conclude that $\mathcal{O}$ must be an element of $Sp(2n+2, \mathbb{R})$ with unit determinant[10]. The new function $\tilde{F}$ is again holomorphic and homogeneous, and follows directly from (C.2),

$$\begin{aligned} \tilde{F}(\tilde{X}) &= \tfrac{1}{2}\tilde{X}^I\, \tilde{F}_I \\ &= i(U^{\rm T}W)_{IJ}X^IX^J + \tfrac{1}{2}(U^{\rm T}V + W^{\rm T}Z)_I{}^J X^I F_J - \tfrac{1}{4}i(Z^{\rm T}V)^{IJ}F_IF_J. \end{aligned} \tag{C.9}$$

Observe that the above result does not correspond to a reparametrization of the function F itself, as this would lead to $\tilde{F}(\tilde{X}) = F(X)$. In this connection it is relevant that the addition of a quadratic polynomial with imaginary coefficients to $F(X)$ does not change the action [5].

We conclude that the Lagrangians (C.1) parametrized by $F(X)$ and $\tilde{F}(\tilde{X})$ represent equivalent theories. Furthermore, when

$$\tilde{F}(\tilde{X}) = F(\tilde{X}), \tag{C.10}$$

[10] In view of the local scale and phase invariance $\mathcal{O}$ may always be multiplied by an arbitrary complex function of the space-time coordinates. We have suppressed these possible multiplicative factors by restricting the determinant of $\mathcal{O}$ to unity.

the Lagrangian is *invariant* under the symplectic transformations. Note that this does *not* imply that F itself is an invariant function. Indeed, from comparison to (C.9) one readily verifies that $F(\tilde{X}) \neq F(X)$, as was already observed in [5] (cf. (2.11)).

The *invariant* reparametrizations, called duality invariances, were studied extensively in [5] in the context of infinitesimal transformations. On the basis of this work it is clear how to extend the reparametrizations to the other fields. Usually they cannot be formulated for the abelian vector fields but are realized on the field strengths by generalized duality transformations [12]. Hence the symplectic parametrizations show that the field equations corresponding to different functions F are equivalent, but not necessarily the actions. Before discussing the symplectic reparametrizations for the other fields, let us introduce some additional notation. Generic $(2n+2)$-dimensional vectors that transform under the action of the $Sp(2n+2,\mathbb{R})$ will be denoted by (u^I, v_J). Under the infinitesimal reparametrizations specified by (2.8), the vector (u^I, v_J) thus transforms according to

$$\begin{aligned} \delta u^I &= B^I{}_J\, u^J - D^{IJ}\, v_J \\ \delta v_I &= C_{IJ}\, u^J - B^J{}_I\, v_J \ . \end{aligned} \tag{C.11}$$

As we have seen, the vector $(X^I, -\frac{1}{2}iF_J)$ is an example of such an $Sp(2n+2,\mathbb{R})$ vector. As we shall discuss below, the spinor and vector fields lead to similar vectors. The symplectic invariant of two $Sp(2n+2,\mathbb{R})$ vectors, (u^I, v_J) and (u'^I, v'_J) takes the form

$$\Omega(u,v;u',v') \equiv u^I\, v'_I - v_I\, u'^I, \tag{C.12}$$

and satisfies $\Omega(\tilde{u},\tilde{v};\tilde{u}',\tilde{v}') = \Omega(u,v;u',v')$.

The transformation of the fields X^I can be written with the aid of a field-dependent matrix $\mathcal{S}(X)$ according to (cf. (C.2))

$$X^I \rightarrow \tilde{X}^I = \mathcal{S}^I{}_J(X)\, X^J, \tag{C.13}$$

where

$$\mathcal{S}^I{}_J(X) \equiv U^I{}_J - \tfrac{1}{2}iZ^{IK}\, F_{KJ} \ . \tag{C.14}$$

Note that this result reduces to (2.10) for infinitesimal transformations as $\mathcal{S}^I{}_J \approx \delta^I_J + B^I{}_J + \frac{1}{2}iD^{IK}F_{KJ}$. It is not difficult to show that $\mathcal{S}$ satisfies the group property of $Sp(2n+2,\mathbb{R})$,

$$\mathcal{S}_1(\tilde{X})\, \mathcal{S}_2(X) = \mathcal{S}_3(X) \ , \tag{C.15}$$

where $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ correspond to the $Sp(2n+2,\mathbb{R})$ matrices $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, with $\mathcal{O}_3 = \mathcal{O}_1\mathcal{O}_2$ and $\tilde{X} = \mathcal{S}_2\, X$. Here use was made of (C.6), which can be written as

$$\tilde{F}_{IJ} = (V_I{}^K F_{KL} + 2iW_{IL})\, (\mathcal{S}^{-1})^L{}_J = (\mathcal{S}^{-1})^L{}_I (F_{LK}\, V_J{}^K + 2iW_{JL}) \ , \tag{C.16}$$

where the right-hand side depends on the coordinates X^I through F_{IJ} and $\mathcal{S}$. From (C.16) it is straightforward to derive the following useful formulae for symplectic reparametrizations of the tensors N_{IJ} and F_{IJK},

$$\begin{aligned} \tilde{N}_{IJ} &= N_{KL}\, (\bar{\mathcal{S}}^{-1})^K{}_I\, (\mathcal{S}^{-1})^L{}_J, \\ \tilde{F}_{IJK} &= F_{MNP}\, (\mathcal{S}^{-1})^M{}_I\, (\mathcal{S}^{-1})^N{}_J\, (\mathcal{S}^{-1})^P{}_K, \end{aligned} \tag{C.17}$$

Given an $Sp(2n{+}2,\mathbb{R})$ vector (u^I, v_J), we may construct (complex) $(n{+}1)$-component vectors by

$$\mathcal{V}_I = v_I + \tfrac{1}{2} i F_{IJ} u^J = N_{IJ} \mathcal{V}^J, \tag{C.18}$$

which transform under $Sp(2n+2,\mathbb{R})$ according to

$$\tilde{\mathcal{V}}_I = \mathcal{V}_J \, (\mathcal{S}^{-1})^J{}_I \ , \qquad \tilde{\mathcal{V}}^I = \bar{\mathcal{S}}^I{}_J \, \mathcal{V}^J \ . \tag{C.19}$$

On the other hand, given a vector $\mathcal{V}^I$ one constructs an $Sp(2n+2,\mathbb{R})$ vector by

$$(u'^I, v'_J) = (\mathcal{V}^I, \tfrac{1}{2} i \bar{F}_{JK} \, \mathcal{V}^K) \ , \tag{C.20}$$

whose imaginary part equals the original vector (u^I, v_J) provided the latter were real.

We now turn to the reparametrizations of the spinor fields Ω_i^I and Ω^{iI} (the index i refers to chiral $SU(2)$; an upper (lower) index denotes the positive (negative) chirality components). Their reparametrizations can be inferred from the fact that both

$$\bar{X}^I \, N_{IJ} \, \Omega_i^J = \bar{X}^I \, (F_{IJ} \, \Omega_i^J) + \bar{F}_I \, \Omega_i^I \tag{C.21}$$

and the kinetic term of the spinors and their coupling to the scalars [22],

$$e^{-1} \mathcal{L}^{\text{fermionic}} = \tfrac{1}{16} \left\{ \bar{\Omega}^{iI} \overleftrightarrow{\not{D}} (F_{IJ} \Omega_i^J) + (\bar{F}_{IJ} \Omega^{iJ}) \overleftrightarrow{\not{D}} \Omega_i^I \right\} \ , \tag{C.22}$$

should preserve their form under symplectic reparametrizations. Both these expressions take the form of the symplectic invariant (C.12) provided we identify $(\Omega_i^I, -\frac{1}{2} i F_{JK} \, \Omega_i^K)$ as an $Sp(2n+2,\mathbb{R})$ vector. This is equivalent to the reparametrization

$$\tilde{\Omega}_i^I = \mathcal{S}^I{}_J \, \Omega_i^J \ . \tag{C.23}$$

What remains are the reparametrizations for the vector fields, which, as emphasized above, are realized on the field strengths. First we evaluate the tensor $\mathcal{N}$ corresponding to the function $\tilde{F}(\tilde{X})$. Denoting this tensor by $\tilde{\mathcal{N}}$, one finds

$$\tilde{\mathcal{N}}_{IK} (U^K{}_J + 2i Z^{KL} \mathcal{N}_{LJ}) = -\tfrac{1}{2} i W_{IJ} + V_I^K \, \mathcal{N}_{KJ} \ . \tag{C.24}$$

As a consistency check, one may multiply both sides of this equation by $\tilde{X}^I$ and X^J; then both sides become proportional to $\tilde{F}(\tilde{X})$ upon using (C.2) and the identities $\mathcal{N}_{IJ} \, X^J = -\frac{1}{4} F_I$ and $F_I \, X^I = 2F$.

On the other hand, on the basis of the results of [5], we expect that the self-dual component of the abelian field strength gives rise to an $Sp(2n{+}2,\mathbb{R})$ vector $(F_{\mu\nu}^{+I}, 2i\mathcal{N}_{JK} \, F_{\mu\nu}^{+K})$. This implies that

$$2i\tilde{\mathcal{N}}_{IJ} \, \tilde{F}_{\mu\nu}^{+I} = 2i\tilde{\mathcal{N}}_{IJ} \, (U + 2iZ\mathcal{N})^J{}_K \, F_{\mu\nu}^{+K} = 2i \, (V\mathcal{N} + W)_{IJ} \ F_{\mu\nu}^{+J} \ . \tag{C.25}$$

which is precisely in accord with the equation (C.24) found above.

After the reduction of the four-dimensional fields we obtain new vectors W_μ^I, A^I and B_I, which are related to the field strengths. From this it follows that $(W_\mu^I, 2i\mathcal{N}_{JK} \, W_\mu^K)$ and (A^I, B_J) are also $Sp(2n+2,\mathbb{R})$ covariant vectors.

The above expressions for the symplectic reparametrizations were all derived and/or used in [5, 19] in infinitesimal form for reparametrizations that constitute an invariance of the theory, i.e., reparametrizations that satisfy (C.10). What we want to stress here is that the restriction to invariance transformations is not necessary. Exactly the same form of the transformations is obtained for arbitrary symplectic reparametrizations.

Finally, when the duality invariances act transitively on the manifold of the spinless fields, transformation rules such as the ones derived above lead to a complete determination of the various functions of the spinless fields that appear in the supergravity Lagrangian, a feature which was extensively exploited in the explicit construction of many supergravity Lagrangians in the past.

For the convenience of the reader we list the $(2n+2)$-component objects transforming under the duality transformations as in (C.2), or infinitesimally as in (C.11), that we encountered in this paper; they are

$$(X^I, -\tfrac{1}{2}iF_J)\ ,\quad (F^{+I}_{\mu\nu}, 2i\mathcal{N}_{JK}\,F^{+K}_{\mu\nu})\ ,\quad (A^I, B_I)\ ,\quad (W^I, 2i\mathcal{N}_{IJ}W^J)\ , \tag{C.26}$$

and their complex conjugates. Furthermore we encountered quantities transforming according to (C.14), such as X^I (cf. (C.13)), the tensors transforming as in (C.17), and $\mathcal{B}_I$ and $\mathcal{B}^I$, defined in (2.32), which transform as $\mathcal{V}_I$ or $\mathcal{V}^I$, respectively (cf. (C.19)).

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